MATHEMATICS

FBISE NOTES

Faderal Board Islamabad

10TH CLASS

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EXERCISE 1.1

1. ***Write the following quadratic equations in the standard form and point out pure quadratic equations.***

(i) $(x + 7)(x - 3) = -7$

Solution: $(x + 7)(x - 3) = -7$

$$x^2 + 4x - 21 = -7$$

$$x^2 + 4x - 21 + 7 = 0$$

$$x^2 + 4x - 14 = 0 \qquad \text{(standard form)}$$

(ii) $\dfrac{x^2 + 4}{3} - \dfrac{x}{7} = 1$

Solution: $\dfrac{x^2 + 4}{3} - \dfrac{x}{7} = 1$

multiplying both sides by 21
(L.C.M. of 3, 7)

$$21\left(\frac{x^2 + 4}{3}\right) - 21\left(\frac{x}{7}\right) = 21(1)$$

$$7\left(x^2 + 4\right) - 3x = 21$$

$$7x^2 + 28 - 3x = 21$$

$$7x^2 + 28 - 3x - 21 = 0$$

$$7x^2 - 3x + 7 = 0 \qquad \text{(standard form)}$$

(iii) $\dfrac{x}{x+1}+\dfrac{x+1}{x}=6$

Solution: $\dfrac{x}{x+1}+\dfrac{x+1}{x}=6$

multiplying both sides by $(x+1)(x)$

$$(x+1)(x)\left(\frac{x}{x+1}\right)+(x+1)(x)\left(\frac{x+1}{x}\right)=6(x+1)(x)$$

$$x^2+(x+1)(x+1)=6(x^2+x)$$

$$x^2+x^2+2x+1=6x^2+6x$$

$$x^2+x^2+2x+1-6x^2-6x=0$$

$$-4x^2-4x+1=0$$

$\rightarrow$ $4x^2+4x-1=0$ (standard form)

Solution: $\dfrac{x+4}{x-2}-\dfrac{x-2}{x}+4=0$

multiplying both sides by $(x-2)(x)$

$$(x-2)(x)\left(\frac{x+4}{x-2}\right)-(x-2)(x)\left(\frac{x-2}{x}\right)+4(x-2)(x)=0$$

$(x-2)(x)$

$$x(x+4)-(x-2)(x-2)+4(x^2-2x)=0$$

$$x^2+4x-(x^2-4x+4)+4x^2-8x=0$$

$$x^2+4x-x^2+4x-4+4x^2-8x=0$$

$4x^2-4=0$ (dividing by 4)

$x^2-1=0$ (pure quadratic equation)

(v) $\dfrac{x+3}{x+4}-\dfrac{x-5}{x}=1$

Solution: $\dfrac{x+3}{x+4}-\dfrac{x-5}{x}=1$

multiplying by $x(x+4)$ both sides.

$$x(x+4)\left(\frac{x+3}{x+4}\right)-x(x+4)\left(\frac{x-5}{x}\right)=x(x+4)(1)$$

$$x(x+3)-(x+4)(x-5)=x^2+4x$$

$$x^2+3x-(x^2-x-20)=x^2+4x$$

$$x^2+3x-x^2+x+20=x^2+4x$$

$$\cancel{x^2}+3x-\cancel{x^2}+x+20-x^2-4x=0$$

$$4x+20-x^2-4x=0$$

$$-x^2+20=0$$

$\Rightarrow \quad x^2-20=0$ (pure quadratic equation)

(vi) $\dfrac{x+1}{x+2}+\dfrac{x+2}{x+3}=\dfrac{25}{12}$

Solution: $\dfrac{x+1}{x+2}+\dfrac{x+2}{x+3}=\dfrac{25}{12}$

multiplying both sides by $12(x+2)(x+3)$

$$12(x+2)(x+3)\left(\frac{x+1}{x+2}\right)+12(x+2)(x+3)\left(\frac{x+2}{x+3}\right)$$

$$=12(x+2)(x+3)\left(\frac{25}{12}\right)$$

$$12(x+3)(x+1)+12(x+2)(x+2)=(x+2)(x+3)25$$

$$12(x^2+4x+3)+12(x^2+4x+4)=25(x^2+5x+6)$$

$$12x^2+48x+36+12x^2+48x+48=25x^2+125x+150$$

$$12x^2+48x+36+12x^2+48x+48-25x^2-125x$$
$$-150=0$$

$$12x^2+12x^2-25x^2+48x+48x-125x+36+48-150=0$$

$$-x^2-29x-66=0$$

$\Rightarrow \quad x^2+29x+66=0$ (standard form)

ختم نبوت ﷺ زندہ باد **عظمت صحابہ زندہ باد**

السلام علیکم ورحمۃ اللہ وبرکاتہ:

معزز ممبران: آپ کا وٹس ایپ گروپ ایڈمن "**اردو بکس**" آپ سے مخاطب ہے۔

آپ تمام ممبران سے گزارش ہے کہ:

- گروپ میں صرف PDF کتب پوسٹ کی جاتی ہیں لہذا کتب کے متعلق اپنے کمنٹس / ریویوز ضرور دیں۔ گروپ میں بغیر ایڈمن کی اجازت کے کسی بھی قسم کی (اسلامی وغیر اسلامی، اخلاقی، تحریری) پوسٹ کرنا سختی سے منع ہے۔
- گروپ میں معزز، پڑھے لکھے، سلجھے ہوئے ممبرز موجود ہیں اخلاقیات کی پابندی کریں اور گروپ رولز کو فالو کریں بصورت دیگر معزز ممبرز کی بہتری کی خاطر ریموو کر دیا جائے گا۔
- کوئی بھی ممبر کسی بھی ممبر کو انباکس میں میسیج، مس کال، کال نہیں کرے گا۔ رپورٹ پر فوری ریموو کر کے کاروائی عمل میں لائے جائے گی۔
- ہمارے کسی بھی گروپ میں سیاسی و فرقہ واریت کی بحث کی قطعاً کوئی گنجائش نہیں ہے۔
- اگر کسی کو بھی گروپ کے متعلق کسی قسم کی شکایت یا تجویز کی صورت میں ایڈمن سے رابطہ کیجئے۔
- سب سے اہم بات:

گروپ میں کسی بھی قادیانی، مرزائی، احمدی، گستاخِ رسول، گستاخِ امہات المؤمنین، گستاخِ صحابہ و خلفائے راشدین حضرت ابو بکر صدیق، حضرت عمر فاروق، حضرت عثمان غنی، حضرت علی المرتضیٰ، حضرت حسنین کریمین رضوان اللہ تعالیٰ اجمعین، گستاخ اہلبیت یا ایسے غیر مسلم جو اسلام اور پاکستان کے خلاف پراپیگنڈا میں مصروف ہیں یا ان کے روحانی و ذہنی سپورٹرز کے لئے کوئی گنجائش نہیں ہے لہذا ایسے اشخاص بالکل بھی گروپ جوائن کرنے کی زحمت نہ کریں۔ معلوم ہونے پر فوراً ریموو کر دیا جائے گا۔

- تمام کتب انٹرنیٹ سے تلاش / ڈاؤنلوڈ کر کے **فری آف کاسٹ** وٹس ایپ گروپ میں شیئر کی جاتی ہیں۔ جو کتاب نہیں ملتی اس کے لئے معذرت کر لی جاتی ہے۔ جس میں محنت بھی صرف ہوتی ہے لیکن ہمیں آپ سے صرف دعاؤں کی درخواست ہے۔
- عمران سیریز کے شوقین کیلئے علیحدہ سے عمران سیریز گروپ موجود ہے۔
- **لیڈیز کے لئے الگ گروپ کی سہولت موجود ہے جس کے لئے ویریفکیشن ضروری ہے۔**
- اردو کتب / عمران سیریز یا سٹڈی گروپ میں ایڈ ہونے کے لئے ایڈمن سے وٹس ایپ پر بذریعہ میسیج رابطہ کریں اور جواب کا انتظار فرمائیں۔ برائے مہربانی اخلاقیات کا خیال رکھتے ہوئے موبائل پر کال یا ایم ایس کرنے کی کوشش ہرگز نہ کریں۔ ورنہ گروپس سے تو ریموو کیا ہی جائے گا بلاک بھی کیا جائے گا۔

نوٹ: ہمارے کسی گروپ کی کوئی فیس نہیں ہے۔ سب فی سبیل اللہ ہے

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اللہ تبارک تعالیٰ ہم سب کا حامی و ناصر ہو

2. ***Solve by factorization:***

(i) $x^2 - x - 20 = 0$

Solution: $x^2 - x - 20 = 0$

$x^2 - 5x + 4x - 20 = 0$

$x(x-5) + 4(x-5) = 0$

$(x-5)(x+4) = 0$

$\Rightarrow \quad x - 5 = 0 \quad or \quad x + 4 = 0$

Means $x = 5$ or $x = -4$

Solution set = $\{5, -4\}$

(ii) $3y^2 = y(y - 5)$

Solution: $3y^2 = y(y-5)$

$3y^2 = y^2 - 5y$

$3y^2 - y^2 + 5y = 0$

$2y^2 + 5y = 0$

$y(2y+5) = 0$

$\Rightarrow \quad y = 0 \quad$ or $\quad 2y + 5 = 0$

Thus $y = 0$ and $2y = -5$

$y = -\dfrac{5}{2}$

Solution set is $\left\{0, -\dfrac{5}{2}\right\}$

(iii) $4 - 32x = 17x^2$

Solution: $4 - 32x = 17x^2$

$4 - 32x - 17x^2 = 0$

$17x^2 + 32x - 4 = 0$

$17x^2 + 34x - 2x - 4 = 0$

$17x(x+2) - 2(x+2) = 0$

$(x+2)(17x-2) = 0$

Thus $x + 2 = 0 \quad or \quad 17x - 2 = 0$

$$\Rightarrow \qquad x = -2, \quad 17x = 2$$

$$x = \frac{2}{17}$$

$$\text{Solution set } = \left\{-2, \frac{2}{17}\right\}$$

(iv) $x^2 - 11x = 152$

Solution: $x^2 - 11x = 152$

$$x^2 - 11x - 152 = 0$$

$$x^2 - 19x + 8x - 152 = 0$$

$$x(x-19) + 8(x-19) = 0$$

$$(x-19)(x+8) = 0$$

Thus, $x - 19 = 0$ ***or*** $x + 8 = 0$

$$x = 19, x = -8$$

Solution Set = {19, −8}

(v) $\dfrac{x+1}{x} + \dfrac{x}{x+1} = \dfrac{25}{12}$

Solution: $\dfrac{x+1}{x} + \dfrac{x}{x+1} = \dfrac{25}{12}$

multiplying both sides by $12x(x+1)$

$$12x(x+1)\left(\frac{x+1}{x}\right) + 12x(x+1)\left(\frac{x}{x+1}\right) = \frac{25}{12} \times 12x(x+1)$$

$$12(x+1)(x+1) + 12x(x) = 25x(x+1)$$

$$12(x^2 + 2x + 1) + 12x^2 = 25(x^2 + x)$$

$$12x^2 + 24x + 12 + 12x^2 = 25x^2 + 25x$$

$$12x^2 + 24x + 12 + 12x^2 - 25x^2 - 25x = 0$$

$$12x^2 + 12x^2 - 25x^2 + 24x - 25x + 12 = 0$$

$$-x^2 - x + 12 = 0$$

or $\qquad x^2 + x - 12 = 0$

$x^2+4x-3x-12=0$

$x(x+4)-3(x+4)=0$

$(x+4)(x-3)=0$

Thus, $x+4=0$ *or* $x-3=0$

$\Rightarrow \quad x=-4,\ x=3$

Solution Set $\{-4, 3\}$

(vi) $\dfrac{2}{x-9}=\dfrac{1}{x-3}-\dfrac{1}{x-4}$

Solution: $\dfrac{2}{x-9}=\dfrac{1}{x-3}-\dfrac{1}{x-4}$

$$\frac{2}{x-9}=\frac{(x-4)-(x-3)}{(x-3)(x-4)}$$

$$\frac{2}{x-9}=\frac{x-4-x+3}{(x-3)(x-4)}$$

$$\frac{2}{x-9}=\frac{-1}{(x-3)(x-4)}$$

multiplying both sides by $(x-9)(x-3)(x-4)$

$$(x-9)(x-3)(x-4)\left(\frac{2}{x-9}\right)=\frac{-1(x-9)(x-3)(x-4)}{(x-3)(x-4)}$$

$2(x-3)(x-4)=-1(x-9)$

$2(x^2-7x+12)=-x+9$

$2x^2-14x+24=-x+9$

$2x^2-14x+24+x-9=0$

$2x^2-14x+x+24-9=0$

$2x^2-13x+15=0$

$2x^2-10x-3x+15=0$

$2x(x-5)-3(x-5)=0$

$(x-5)(2x-3)=0$

Thus, $x-5=0$ *or* $2x-3=0$

$x=5$, $2x=3$

$x=\frac{3}{2}$

Solution set $=\left\{5, \frac{3}{2}\right\}$

3. Solve the following equations by completing square:

(i) $7x^2+2x-1=0$

Solution: $7x^2+2x-1=0$

Dividing by 7, both sides

$$x^2+\frac{2}{7}x-\frac{1}{7}=0$$

$$x^2+\frac{2}{7}x=\frac{1}{7}$$

Adding $\left[\frac{1}{2}\left(\frac{2}{7}\right)\right]^2=\left(\frac{1}{7}\right)^2$ on both sides

$$x^2+\frac{2}{7}x+\left(\frac{1}{7}\right)^2=\frac{1}{7}+\left(\frac{1}{7}\right)^2$$

$$\left(x+\frac{1}{7}\right)^2=\frac{1}{7}+\frac{1}{49}$$

$$x+\frac{1}{7}=\frac{7+1}{49}$$

$$\left(x+\frac{1}{7}\right)^2=\frac{8}{49}$$

Taking square root of both sides

$$x+\frac{1}{7}=\pm\sqrt{\frac{8}{49}}$$

$$x+\frac{1}{7}=\pm\frac{2\sqrt{2}}{7}$$

$$x=-\frac{1}{7}\pm\frac{2\sqrt{2}}{7}$$

$$x=\frac{-1\pm2\sqrt{2}}{7}$$

$$\text{Solution set }=\left\{\frac{-1\pm2\sqrt{2}}{7}\right\}$$

(ii) $\boldsymbol{ax^2+4x-a=0,\ a\neq0}$

Solution: $ax^2+4x-a=0$

$$ax^2+4x=a$$

Dividing by both sides

$$x^2+\frac{4}{a}x=1$$

Adding $\left[\frac{1}{2}\left(\frac{4}{a}\right)\right]^2=\left(\frac{2}{a}\right)^2$ on both sides

$$x^2+\frac{4}{a}x+\left(\frac{2}{a}\right)^2=1+\left(\frac{2}{a}\right)^2$$

$$\left(x+\frac{2}{a}\right)^2=1+\frac{4}{a^2}$$

$$\left(x+\frac{2}{a}\right)^2=\frac{a^2+4}{a^2}$$

Taking square root of both sides

$$x+\frac{2}{a}=\pm\frac{\sqrt{a^2+4}}{a}$$

$$x = -\frac{2}{a} \pm \frac{\sqrt{a^2+4}}{a}$$

$$x = \frac{-2 \pm \sqrt{a^2+4}}{a}$$

$$\text{Solution set} = \left\{\frac{-2 \pm \sqrt{a^2+4}}{a}\right\}$$

(iii) $11x^2 - 34x + 3 = 0$

Solution: $11x^2 - 34x + 3 = 0$

$$11x^2 - 34x = -3$$

Dividing by 11, both sides

$$x^2 - \frac{34}{11}x = -\frac{3}{11}$$

Adding $\left[\frac{1}{2}\left(-\frac{34}{11}\right)\right]^2 = \left(-\frac{17}{11}\right)^2$ on both sides

$$x^2 - \frac{34}{11}x + \left(-\frac{17}{11}\right)^2 = -\frac{3}{11} + \left(-\frac{17}{11}\right)^2$$

$$\left(x - \frac{17}{11}\right)^2 = -\frac{3}{11} + \frac{289}{121}$$

$$\left(x - \frac{17}{11}\right)^2 = \frac{-33+289}{121}$$

$$\left(x - \frac{17}{11}\right)^2 = \frac{256}{121}$$

Taking square root of both sides

$$x - \frac{17}{11} = \pm\sqrt{\frac{256}{121}}$$

$$x-\frac{17}{11}=\pm\frac{16}{11}$$

$$x=\frac{17}{11}\pm\frac{16}{11}$$

Either

$$x=\frac{17}{11}+\frac{16}{11} \quad or \quad x=\frac{17}{11}-\frac{16}{11}$$

$$x=\frac{17+16}{11} \quad , \quad x=\frac{17-16}{11}$$

$$x=\frac{33}{11} \quad , \quad x=\frac{1}{11}$$

$$x=3$$

$$\text{Solution set }=\left\{3,\ \frac{1}{11}\right\}$$

(iv) $lx^2+mx+n=0,\ l\neq 0$

Solution: $lx^2+mx+n=0$

$$lx^2+mx=-n$$

Dividing both sides by l.

$$x^2+\frac{m}{l}x=-\frac{n}{l}$$

Adding $\left[\frac{1}{2}\left(\frac{m}{2l}\right)\right]^2=\left(\frac{m}{l}\right)^2$ on both sides

$$x^2+\frac{m}{l}x+\left(\frac{m}{2l}\right)^2=-\frac{n}{l}+\left(\frac{m}{2l}\right)^2$$

$$\left(x+\frac{m}{2l}\right)^2=-\frac{n}{l}+\frac{m^2}{4l^2}$$

$$\left(x+\frac{m}{2l}\right)^2=\frac{-4nl+m^2}{4l^2}$$

$$\left(x+\frac{m}{2l}\right)^2=\frac{m^2-4nl}{4l^2}$$

Taking square root of both sides

$$x+\frac{m}{2l}=\pm\frac{\sqrt{m^2-4nl}}{2l}$$

$$x=-\frac{m}{2l}\pm\frac{\sqrt{m^2-4nl}}{2l}$$

$$x=\frac{-m\pm\sqrt{m^2-4nl}}{2l}$$

$$\text{Solution set}=\left\{\frac{-m\pm\sqrt{m^2-4nl}}{2l}\right\}$$

(v) $3x^2+7x=0$

Solution: $3x^2+7x=0$

Dividing by 3, both sides

$$x^2+\frac{7}{3}x=0$$

Adding $\left[\frac{1}{2}\left(\frac{7}{3}\right)\right]^2=\left(\frac{7}{6}\right)^2$ on both sides

$$x^2+\frac{7}{3}x+\left(\frac{7}{6}\right)^2=0+\left(\frac{7}{6}\right)^2$$

$$\left(x+\frac{7}{6}\right)^2=\left(\frac{7}{6}\right)^2$$

Taking square root on both sides

$x+\frac{7}{6}=\pm\frac{7}{6}$

Either

$x+\frac{7}{6}=\frac{7}{6}$ or $x+\frac{7}{6}=\frac{-7}{6}$

$x=\frac{7}{6}-\frac{7}{6}$, $x=\frac{-7}{6}-\frac{7}{6}$

$x=0$, $x=\frac{-7-7}{6}$

$x=\frac{-14}{6}$

$x=-\frac{7}{3}$

Sol. set $=\left\{0,-\frac{7}{3}\right\}$

(vi) $x^2-2x-195=0$

Solution: $x^2-2x-195=0$

$x^2-2x=195$

Adding $\left[\frac{1}{2}(-2)\right]^2=(-1)^2$ on both sides

$x^2-2x+(-1)^2=195+(-1)^2$

$(x-1)^2=195+1$

$(x-1)^2=196$

Taking square root of both sides

$x-1=\pm\sqrt{196}$

$x-1=\pm 14$

Either

$x-1=14$ or $x-1=-14$

$x = 14 + 1 \quad , \quad x = -14 + 1$

$x = 15 \quad , \quad x = -13$

Solution set = {15, −13}

(vii) $-x^2 + \frac{15}{2} = \frac{7}{2}x$

Solution: $-x^2 + \frac{15}{2} = \frac{7}{2}x$

multiplying by 2, both sides

$-2x^2 + 15 = 7x$

$\Rightarrow \quad 2x^2 + 7x = 15$

Dividing by 2, both sides

$x^2 + \frac{7}{2}x = \frac{15}{2}$

Adding $\left[\frac{1}{2}\left(\frac{7}{2}\right)\right]^2 = \left(\frac{7}{4}\right)^2$ on both sides

$x^2 + \frac{7}{2}x + \left(\frac{7}{4}\right)^2 = \frac{15}{2} + \left(\frac{7}{4}\right)^2$

$\left(x + \frac{7}{4}\right)^2 = \frac{15}{2} + \frac{49}{16}$

$\left(x + \frac{7}{4}\right)^2 = \frac{120 + 49}{16}$

$\left(x + \frac{7}{4}\right)^2 = \frac{169}{16}$

Taking square root of both sides

$x + \frac{7}{4} = \pm\sqrt{\frac{169}{16}}$

$$x+\frac{7}{4}=\pm\frac{13}{4}$$

Either

$$x+\frac{7}{4}=\frac{13}{4} \quad or \quad x+\frac{7}{4}=-\frac{13}{4}$$

$$x=\frac{13}{4}-\frac{7}{4} \quad , \quad x=-\frac{13}{4}-\frac{7}{4}$$

$$x=\frac{13-7}{4} \quad , \quad x=\frac{-13-7}{4}$$

$$x=\frac{6}{4} \quad , \quad x=-\frac{20}{4}$$

$$x=\frac{3}{2} \quad , \quad x=-5$$

$$\text{Solution set}=\left\{\frac{3}{2},-5\right\}$$

(viii) $x^2+17x+\frac{33}{4}=0$

Solution: $x^2+17x+\frac{33}{4}=0$

$$x^2+17x=-\frac{33}{4}$$

Adding $\left[\frac{1}{2}(17)\right]^2=\left(\frac{17}{2}\right)^2$ on both sides

$$x^2+17x+\left(\frac{17}{2}\right)^2=-\frac{33}{4}+\left(\frac{17}{2}\right)^2$$

$$\left(x+\frac{17}{2}\right)^2=-\frac{33}{4}+\frac{289}{4}$$

$$\left(x+\frac{17}{2}\right)^2=\frac{-33+289}{4}$$

$$\left(x+\frac{17}{2}\right)^2=\frac{256}{4}$$

Taking square root of both sides

$$x+\frac{17}{2}=\pm\sqrt{\frac{256}{4}}$$

$$x+\frac{17}{2}=\pm\frac{16}{2}$$

Either

$$x+\frac{17}{2}=\frac{16}{2} \quad or \quad x+\frac{17}{2}=-\frac{16}{2}$$

$$x=\frac{16}{2}-\frac{17}{2} \quad , \quad x=-\frac{16}{2}-\frac{17}{2}$$

$$x=\frac{16-17}{2} \quad , \quad x=\frac{-16-17}{2}$$

$$x=-\frac{1}{2} \quad , \quad x=-\frac{33}{2}$$

Solution set $=\left\{-\frac{1}{2},-\frac{33}{2}\right\}$

(ix) $4-\frac{8}{3x+1}=\frac{3x^2+5}{3x+1}$

Solution: $4-\frac{8}{3x+1}=\frac{3x^2+5}{3x+1}$

multiplying by $3x+1$, both sides

$$4(3x+1)-(3x+1)\left(\frac{8}{3x+1}\right)=(3x+1)\left(\frac{3x^2+5}{3x+1}\right)$$

$$12x+4-8=3x^2+5$$

$12x - 4 = 3x^2 + 5$
$3x^2 - 12x + 5 + 4 = 0$
$3x^2 - 12x + 9 = 0$
Dividing by 3, both sides
$x^2 - 4x + 3 = 0$
$x^2 - 4x = -3$
Adding $\left[\frac{1}{2}(-4)\right]^2 = (-2)^2$ on both sides
$x^2 - 4x + (-2)^2 = -3 + (-2)^2$
$(x-2)^2 = -3 + 4$
$(x-2)^2 = 1$
Taking square root of both sides
Either

$x - 2 = 1$	$x - 2 = -1$
$x = 1 + 2$	$x = -1 + 2$
$x = 3$	$x = 1$

Solution set = $\{3, 1\}$

(x) $7(x + 2a)^2 + 3a^2 = 5a(7x + 23a)$

Solution: $7(x + 2a)^2 + 3a^2 = 5a(7x + 23a)$
$7(x^2 + 4ax + 4a^2) + 3a^2 = 35ax + 115a^2$
$7x^2 + 28ax + 28a^2 + 3a^2 = 35ax + 115a^2$
$7x^2 + 28ax - 35ax + 31a^2 - 115a^2 = 0$
$7x^2 - 7ax - 84a^2 = 0$
Dividing by 7, both sides
$x^2 - ax - 12a^2 = 0$
$x^2 - ax = 12a^2$
Adding $\left[\frac{1}{2}(-a)\right]^2 = \left(\frac{-a}{2}\right)^2$ on both sides

$$x^2 - ax + \left(\frac{-a}{2}\right)^2 = 12a^2 + \left(\frac{-a}{2}\right)^2$$

$$\left(x - \frac{a}{2}\right)^2 = 12a^2 + \frac{a^2}{4}$$

$$\left(x - \frac{a}{2}\right)^2 = \frac{48a^2 + a^2}{4}$$

$$\left(x - \frac{a}{2}\right)^2 = \frac{49a^2}{4}$$

Taking square root of both sides

$$x - \frac{a}{2} = \pm\sqrt{\frac{49a^2}{4}}$$

$$x - \frac{a}{2} = \pm\frac{7a}{2}$$

Either

$$x - \frac{a}{2} = \frac{7a}{2} \quad or \quad x - \frac{a}{2} = \frac{-7a}{2}$$

$$x = \frac{7a}{2} + \frac{a}{2} \quad , \quad x = \frac{-7a}{2} + \frac{a}{2}$$

$$x = \frac{7a + a}{2} \quad , \quad x = \frac{-7a + a}{2}$$

$$x = \frac{8a}{2} \quad , \quad x = \frac{-6a}{2}$$

$$x = 4a \quad , \quad x = -3a$$

Solution set = $\{4a, -3a\}$

Quadratic Formula

If $ax^2 + bx + c = 0$

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

EXERCISE 1.2

1. *Solve the following equations using quadratic formula:*

(i) $2 - x^2 = 7x$

Solution: $2 - x^2 = 7x$

$$-x^2 - 7x + 2 = 0$$

$$x^2 + 7x - 2 = 0$$

Here $a = 1, b = 7, c = -2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Putting values of a, b, c. we get

$$x = \frac{-(7) \pm \sqrt{(7)^2 - 4(1)(-2)}}{2(1)}$$

$$x = \frac{-7 \pm \sqrt{49 + 8}}{2}$$

$$x = \frac{-7 \pm \sqrt{57}}{2}$$

Solution set = $\left\{\frac{-7 \pm \sqrt{57}}{2}\right\}$

(ii) $5x^2 + 8x + 1 = 0$

Solution: $5x^2 + 8x + 1 = 0$

Here, $a = 5,\ b = 8,\ c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Putting values of a, b, c, we get

$$x=\frac{-(8)\pm\sqrt{(8)^2-4(5)(1)}}{2(5)}$$

$$x=\frac{-8\pm\sqrt{64-20}}{10}$$

$$x=\frac{-8\pm\sqrt{44}}{10}$$

$$x=\frac{-8\pm\sqrt{2\times2\times11}}{10}$$

$$x=\frac{-8\pm2\sqrt{11}}{10}$$

$$x=\frac{2\left(-4\pm\sqrt{11}\right)}{10}$$

$$x=\frac{-4\pm\sqrt{11}}{5}$$

Solution set $=\left\{\frac{-4\pm\sqrt{11}}{5}\right\}$

(iii) $\sqrt{3}\,x^2+x=4\sqrt{3}$

Solution: $\sqrt{3}\,x^2+x=4\sqrt{3}$

$$\sqrt{3}x^2+x-4\sqrt{3}=0$$

Here; $a=\sqrt{3},\ b=1,\ c=-4\sqrt{3}$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Putting values of *a, b, c* we get

$$x=\frac{-1\pm\sqrt{(1)^2-4\left(\sqrt{3}\right)\left(-4\sqrt{3}\right)}}{2\sqrt{3}}$$

$$x=\frac{-1\pm\sqrt{1+16\times 3}}{2\sqrt{3}}$$

$$x=\frac{-1\pm\sqrt{49}}{2\sqrt{3}}$$

$$x=\frac{-1\pm 7}{2\sqrt{3}}$$

$$x=\frac{-1+7}{2\sqrt{3}} \quad or \quad x=\frac{-1-7}{2\sqrt{3}}$$

$$x=\frac{6}{2\sqrt{3}} \quad , \quad \frac{-8}{2\sqrt{3}}$$

$$x=\frac{3}{\sqrt{3}} \quad , \quad \frac{-4}{\sqrt{3}}$$

$$x=\sqrt{3}$$

Solution set $=\left\{\sqrt{3},\ \frac{-4}{\sqrt{3}}\right\}$

(iv) $4x^2-14=3x$

Solution: $4x^2-14=3x$

$4x^2-3x-14=0$

Here, $a=4, b=-3, c=-14$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Putting values of a, b, c, we get

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(-14)}}{2(4)}$$

$$x = \frac{3 \pm \sqrt{9+224}}{8}$$

$$x = \frac{3 \pm \sqrt{233}}{8}$$

Solution set $= \left\{\frac{3 \pm \sqrt{233}}{8}\right\}$

(v) $6x^2 - 3 - 7x = 0$

Solution: $6x^2 - 3 - 7x = 0$

$$6x^2 - 7x - 3 = 0$$

Here, $a = 6, b = -7, c = -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Putting values of a, b, c, we get

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(6)(-3)}}{2(6)}$$

$$x = \frac{7 \pm \sqrt{49+72}}{12}$$

$$x = \frac{7 \pm \sqrt{121}}{12}$$

$$x = \frac{7 \pm 11}{12}$$

$$x = \frac{7+11}{12}, \frac{7-11}{12}$$

$$x = \frac{18}{12}, \frac{-4}{12}$$

$$x = \frac{3}{2}, \frac{-1}{3}$$

$$\text{Solution set} = \left\{\frac{3}{2}, -\frac{1}{3}\right\}$$

(vi) $3x^2 + 8x + 2 = 0$

Solution: $3x^2 + 8x + 2 = 0$

Here, $a = 3, b = 8, c = 2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Putting values of *a, b, c,* we get

$$x = \frac{-8 \pm \sqrt{(8)^2 - 4(3)(2)}}{2(3)}$$

$$x = \frac{-8 \pm \sqrt{64 - 24}}{6}$$

$$x = \frac{-8 \pm \sqrt{40}}{6}$$

$$x = \frac{-8 \pm 2\sqrt{10}}{6}$$

$$x = \frac{2\left[-4 \pm \sqrt{10}\right]}{6}$$

$$x = \frac{-4 \pm \sqrt{10}}{3}$$

$$\text{Solution set} = \left\{\frac{-4 \pm \sqrt{10}}{3}\right\}$$

(vii) $\frac{3}{x-6}-\frac{4}{x-5}=1$

Solution: $\frac{3}{x-6}-\frac{4}{x-5}=1$

multiplying by $(x-6)(x-5)$ both sides

$$(x-6)(x-5)\left(\frac{3}{x-6}\right)-(x-6)(x-5)\left(\frac{4}{x-5}\right)$$

$$=1(x-6)(x-5)$$

$$3(x-5)-4(x-6)=(x-6)(x-5)$$

$$3x-15-4x+24=x^2-11x+30$$

$$9-x=x^2-11x+30$$

$$0=x^2-11x+x+30-9$$

$$0=x^2-10x+21$$

or $x^2-10x+21=0$

Here, $a=1, b=-10, c=21$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

putting values of a, b, c, we get

$$x=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(21)}}{2(1)}$$

$$x=\frac{10\pm\sqrt{100-84}}{2}$$

$$x=\frac{10\pm\sqrt{16}}{2}$$

$$x=\frac{10\pm4}{2} \quad OR$$

$$x=\frac{10+4}{2},\frac{10-4}{2}$$

$$x = \frac{14}{2} \quad , \quad \frac{6}{2}$$

$$x = 7, 3$$

Solution set = {7, 3}

(viii) $\frac{x+2}{x-1} - \frac{4-x}{2x} = 2\frac{1}{3}$

Solution: $\frac{x+2}{x-1} - \frac{4-x}{2x} = 2\frac{1}{3}$

$$\frac{x+2}{x-1} - \frac{4-x}{2x} = \frac{7}{3}$$

Multiplying by $3(x-1)(2x)$, we get

$$3(x-1)(2x)\left(\frac{x+2}{x-1}\right) - 3(x-1)(2x)\left(\frac{4-x}{2x}\right)$$

$$= \frac{7}{3} \times 3(x-1)(2x)$$

$$3(2x)(x+2) - 3(x-1)(4-x) = 7(x-1)(2x)$$

$$6x(x+2) - (3x-3)(4-x) = 14x(x-1)$$

$$6x^2 + 12x - (12x - 3x^2 - 12 + 3x) = 14x^2 - 14x$$

$$6x^2 + 12x - 12x + 3x^2 + 12 - 3x = 14x^2 - 14x$$

$$9x^2 - 3x + 12 - 14x^2 + 14x = 0$$

$$-5x^2 + 11x + 12 = 0$$

$$5x^2 - 11x - 12 = 0$$

Here, $a = 5, b = -11, c = -12$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Putting values of a, b, c, we get

$$x=\frac{-(-11)\pm\sqrt{(-11)^2-4(5)(-12)}}{2(5)}$$

$$x=\frac{11\pm\sqrt{121+240}}{10}$$

$$x=\frac{11\pm\sqrt{361}}{10}$$

$$x=\frac{11\pm19}{10}$$

$$x=\frac{11+19}{10} \quad or \quad \frac{11-19}{10}$$

$$x=\frac{30}{10} \quad , \quad \frac{-8}{10}$$

$$x=3 \quad , -\frac{4}{5}$$

Solution set $=\left\{3,\ -\frac{4}{5}\right\}$

(ix) $\dfrac{a}{x-b}+\dfrac{b}{x-a}=2$

Solution: $\dfrac{a}{x-b}+\dfrac{b}{x-a}=2$

Multiplying by $(x-b)(x-a)$ both sides

$$(x-b)(x-a)\left(\frac{a}{x-b}\right)+(x-b)(x-a)\left(\frac{b}{x-a}\right)$$

$$=2(x-b)(x-a)$$

$$a(x-a)+b(x-b)=2(x^2-ax-bx+ab)$$

$$ax-a^2+bx-b^2=2x^2-2ax-2bx+2ab$$

$$-2x^2+ax+bx+2ax+2bx-2ab-a^2-b^2=0$$

$$-2x^2+3ax+3bx-2ab-a^2-b^2=0$$
$$2x^2-3ax-3bx+2ab+a^2+b^2=0$$
$$2x^2-(3a+3b)x+(a+b)^2=0$$

Here, $a'=2,\ b'=-(3a+3b),\ c'=(a+b)^2$

$$x=\frac{-b\pm\sqrt{(b)^2-4ac}}{2a}$$

Putting values of *a, b, c* we get

$$x=\frac{[-(3a+3b)]\pm\sqrt{[-(3a+3b)]^2-4(2)(a+b)^2}}{2(2)}$$
$$x=\frac{(3a+3b)\pm\sqrt{9a^2+18ab+9b^2-8(a^2+2ab+b^2)}}{4}$$
$$x=\frac{(3a+3b)\pm\sqrt{9a^2+18ab+9b^2-8a^2-16ab-8b^2}}{4}$$
$$x=\frac{(3a+3b)\pm\sqrt{a^2+2ab+b^2}}{4}$$
$$x=\frac{(3a+3b)\pm\sqrt{(a+b)^2}}{4}$$
$$x=\frac{(3a+3b)\pm(a+b)}{4}$$
$$x=\frac{(3a+3b)+(a+b)}{4}\quad or\quad x=\frac{(3a+3b)-(a+b)}{4}$$
$$x=\frac{3a+3b+a+b}{4}\ ,\quad x=\frac{3a+3b-a-b}{4}$$
$$x=\frac{4a+4b}{4}\ ,\quad x=\frac{2a+2b}{4}$$
$$x=\frac{4(a+b)}{4}\ ,\quad x=\frac{2(a+b)}{4}$$
$$x=a+b\ ,\quad \frac{a+b}{2}$$

Solution set $= \left\{a+b,\ \dfrac{a+b}{2}\right\}$

(x) $-(l+m)-lx^2+(2l+m)x=0, l\neq 0$

Solution: $-(l+m)-lx^2+(2l+m)x=0$

$\Rightarrow \quad lx^2-(2l+m)x+(l+m)=0$

Here, $a=l, b=-(2l+m), c=(l+m)$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Putting values of a, b, c, we get.

$$x=\frac{-\left[-(2l+m)\right]\pm\sqrt{\left[-(2l+m)\right]^2-4l(l+m)}}{2l}$$

$$x=\frac{(2l+m)\pm\sqrt{4l^2+4lm+m^2-4l^2-4lm}}{2l}$$

$$x=\frac{(2l+m)\pm\sqrt{m^2}}{2l}$$

$$x=\frac{(2l+m)\pm m}{2l}$$

$$x=\frac{2l+m+m}{2l} \quad , \quad x=\frac{2l+m-m}{2l}$$

$$x=\frac{2l+2m}{2l} \quad , \quad x=\frac{2l}{2l}$$

$$x=\frac{2(l+m)}{2l} \quad , \quad x=1$$

$$x=\frac{l+m}{l} \quad , \quad x=1$$

Solution set $= \left\{\dfrac{l+m}{l},\ 1\right\}$

EXERCISE 1.3

Solve the following equations.

1. $2x^4 - 11x^2 + 5 = 0$

Solution: $2x^4 - 11x^2 + 5 = 0$

Let $x^2 = y$, then $x^4 = y^2$, the given equation becomes

$$2y^2 - 11y + 5 = 0$$
$$2y^2 - 10y - y + 5 = 0$$
$$2y(y-5) - 1(y-5) = 0$$
$$(2y-1)(y-5) = 0$$

$2y \quad 1 \quad 0$ or $y - 5 = 0$

$2y \quad 1$ $\qquad y = 5$

$y \quad \dfrac{1}{2}$

If $y \quad \dfrac{1}{2}$	If $y = 5$
then $x^2 - \dfrac{1}{2}$	then $x^2 = 5$
Thus, $x = \pm\dfrac{1}{\sqrt{2}}$	Thus, $x = \pm\sqrt{5}$

$$\text{Sol. set} = \left\{\pm\frac{1}{\sqrt{2}}, \pm\sqrt{5}\right\}$$

2. $2x^4 = 9x^2 - 4$

Solution: $2x^4 = 9x^2 - 4$

$$2x^4 - 9x^2 + 4 = 0$$
$$2x^4 - x^2 - 8x^2 + 4 = 0$$
$$x^2(2x^2 - 1) - 4(2x^2 - 1) = 0$$

$(2x^2-1)(x^2-4)=0$

$2x^2-1=0$ or $x^2-4=0$

$2x^2=1$ | $x^2=4$

$x^2=\frac{1}{2}$ | $x=\pm 2$

$x=\pm\frac{1}{\sqrt{2}}$

Sol. set $=\left\{\pm\frac{1}{\sqrt{2}}, \pm 2\right\}$

3. $5x^{1/2}=7x^{1/4}-2$

Solution: $5x^{1/2}=7x^{1/4}-2$

$5x^{1/2}-7x^{1/4}+2=0$

Let $x^{\frac{1}{4}}=y$ then $x^{\frac{1}{2}}=y^2$, the given Equation. becomes

$5y^2-7y+2=0$

$5y^2-5y-2y+2=0$

$5y(y-1)-2(y-1)=0$

$(y-1)(5y-2)=0$

$y-1=0$ or $5y-2=0$

$y=1$ | $5y=2$

$y=\frac{2}{5}$

If $y=1$ | If $y=\frac{2}{5}$

then $x^{\frac{1}{4}}=1$ | then $x^{\frac{1}{4}}=\frac{2}{5}$

$$\left(x^{\frac{1}{4}}\right)^4 = (1)^4 \qquad \left(x^{\frac{1}{4}}\right)^4 = \left(\frac{2}{5}\right)^4$$

$$x = 1 \qquad x = \frac{16}{625}$$

$$\text{Sol. set} = \left\{1, \frac{16}{625}\right\}$$

4. $x^{2/3} + 54 = 15x^{1/3}$

Solution: $x^{2/3} + 54 = 15x^{1/3}$

$$x^{\frac{2}{3}} - 15x^{\frac{1}{3}} + 54 = 0$$

Let $x^{\frac{1}{3}} = y$ then $x^{\frac{2}{3}} = y^2$, the given equation becomes

$$y^2 - 15y + 54 = 0$$

$$y^2 - 9y - 6y + 54 = 0$$

$$y(y-9) - 6(y-9) = 0$$

$$(y-9)(y-6) = 0$$

$$y - 9 = 0 \quad \text{or} \quad y - 6 = 0$$

$$y = 9 \quad \text{or} \quad y = 6$$

If $y = 9$

then $x^{\frac{1}{3}} = 9$

$$\left(x^{\frac{1}{3}}\right)^3 = (9)^3$$

$$x = 729$$

If $y = 6$

then $x^{\frac{1}{3}} = 6$

$$\left(x^{\frac{1}{3}}\right)^3 = (6)^3$$

$$x = 216$$

$$\text{Sol. set} = \{729, 216\}$$

5. $3x^{-2} + 5 = 8x^{-1}$

Solution: $3x^{-2} + 5 = 8x^{-1}$

$$3x^{-2} - 8x^{-1} + 5 = 0$$

Let $x^{-1} = y$ then $x^{-2} = y^2$, the given equation becomes

$$3y^2 - 8y + 5 = 0$$
$$3y^2 - 3y - 5y + 5 = 0$$
$$3y(y-1) - 5(y-1) = 0$$
$$(y-1)(3y-5) = 0$$

$y - 1 = 0$ or $3y - 5 = 0$

$y = 1$ | $3y = 5$

If $y = 1$ | $y = \frac{5}{3}$

then $x^{-1} = 1$ | If $y = \frac{5}{3}$

$\frac{1}{x} = 1$ | then $x^{-1} = \frac{5}{3}$

$x = 1$ | $x = \frac{3}{5}$

Sol. set $= \left\{1, \frac{3}{5}\right\}$

6. $(2x^2 + 1) + \frac{3}{2x^2 + 1} = 4$

Solution: $(2x^2 + 1) + \frac{3}{2x^2 + 1} = 4$

Let $2x^2 + 1 = y$, the given equation becomes

$y + \frac{3}{y} = 4$ (multiplying by y)

$\Rightarrow$ $y^2 + 3 = 4y$

$$y^2 - 4y + 3 = 0$$
$$y^2 - y - 3y + 3 = 0$$

$y(y-1)-3(y-1)=0$

$(y-1)(y-3)=0$

$y-1=0$ or $y-3=0$

$y=1$ | $y=3$

If $y=1$

then $2x^2+1=1$

$2x^2=1-1$

$2x^2=0$

$x=0$

If $y=3$

then $2x^2+1=3$

$2x^2=3-1$

$2x^2=2$

$x^2=1$

$x=\pm 1$

Sol. set $=\{0, \pm 1\}$

7. $\frac{x}{x-3}+4\left(\frac{x-3}{x}\right)=4$

Solution: $\frac{x}{x-3}+4\left(\frac{x-3}{x}\right)=4$

Let $\frac{x}{x-3}=y$, then $\frac{x-3}{x}=\frac{1}{y}$, the given equation becomes

$y+\frac{4}{y}=4$ (multiplying by y)

$y^2+4=4y$

$y^2-4y+4=0$

$(y-2)^2=0$

$y-2=0$

$y=2$

When $y=2$

then $\frac{x}{x-3}=2$

$x = 2(x-3)$
$x = 2x - 6$
$x - 2x = -6$
$-x = -6$
$x = 6$
Solution set = {6}

8. $\frac{4x+1}{4x-1} + \frac{4x-1}{4x+1} = 2\frac{1}{6}$

Solution: $\frac{4x+1}{4x-1} + \frac{4x-1}{4x+1} = 2\frac{1}{6}$

Let $\frac{4x+1}{4x-1} = y$, then $\frac{4x-1}{4x+1} = \frac{1}{y}$, the given equation becomes

$y + \frac{1}{y} = \frac{13}{6}$ (multiplying by $6y$)

$6y^2 + 6 = 13y$
$6y^2 - 13y + 6 = 0$
$6y^2 - 4y - 9y + 6 = 0$
$2y(3y-2) - 3(3y-2) = 0$
$(3y-2)(2y-3) = 0$

$3y - 2 = 0$ or $2y - 3 = 0$

$3y = 2$	$2y = 3$
$y = \frac{2}{3}$	$y = \frac{3}{2}$
If $y = \frac{2}{3}$	If $y = \frac{3}{2}$

then $\frac{4x+1}{4x-1}=\frac{2}{3}$

$3(4x+1)=2(4x-1)$

$12x+3=8x-2$

$12x-8x=-2-3$

$4x=-5$

$x=\frac{-5}{4}$

then $\frac{4x+1}{4x-1}=\frac{3}{2}$

$2(4x+1)=3(4x-1)$

$8x+2=12x-3$

$12x-8x=2+3$

$4x=5$

$x=\frac{5}{4}$

Sol. set $=\left\{-\frac{5}{4},\ \frac{5}{4}\right\}$

9. $\frac{x-a}{x+a}-\frac{x+a}{x-a}=\frac{7}{12}$

Solution: $\frac{x-a}{x+a}-\frac{x+a}{x-a}=\frac{7}{12}$

Let $\frac{x-a}{x+a}=y$, then $\frac{x+a}{x-a}=\frac{1}{y}$, the given equation becomes

$y-\frac{1}{y}=\frac{7}{12}$ (multiplying by $12y$)

$12y^2-12=7y$

$12y^2-7y-12=0$

$12y^2-16y+9y-12=0$

$4y(3y-4)+3(3y-4)=0$

$(3y-4)(4y+3)=0$

$3y-4=0$ or $4y+3=0$

$3y=4$ | $4y=-3$

$y=\frac{4}{3}$ | $y=-\frac{3}{4}$

when $y = \frac{4}{3}$

then $\frac{x-a}{x+a} = \frac{4}{3}$

$3(x-a) = 4(x+a)$

$3x - 3a = 4x + 4a$

$3x - 4x = 4a + 3a$

$-x = 7a$

$x = -7a$

when $y = -\frac{3}{4}$

then $\frac{x-a}{x+a} = -\frac{3}{4}$

$4(x-a) = -3(x+a)$

$4x - 4a = -3x - 3a$

$4x + 3x = -3a + 4a$

$7x = a$

$x = \frac{a}{7}$

Sol. set $= \left\{-7a, \frac{a}{7}\right\}$

10. $x^4 - 2x^3 - 2x^2 + 2x + 1 = 0$

Solution: $x^4 - 2x^3 - 2x^2 + 2x + 1 = 0$

Dividing by x^2, we get

$x^2 - 2x - 2 + \frac{2}{x} + \frac{1}{x^2} = 0$

regrouping

$\left(x^2 + \frac{1}{x^2}\right) - 2\left(x - \frac{1}{x}\right) - 2 = 0$

Let $x - \frac{1}{x} = y$ (squaring)

$x^2 + \frac{1}{x^2} - 2 = y^2$

$x^2 + \frac{1}{x^2} = y^2 + 2$ the given equation becomes

$y^2 + 2 - 2y - 2 = 0$

$y^2 - 2y = 0$

$$y(y-2)=0$$

$y=0$ or $y-2=0$

$y=2$

when $y=0$	when $y=2$
$x-\frac{1}{x}=0$	$x-\frac{1}{x}=2$
(Multiplying by x)	(Multiplying by x)
$x^2-1=0$	
$x^2=1$	$x^2-1=2x$
$x=\pm 1$	$x^2-2x-1=0$
	Here, $a=1,\ b=-2,\ c=-1$
	$x=\frac{-b\pm\sqrt{(b)^2-4ac}}{2a}$
	$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-1)}}{2(1)}$
	$x=\frac{2\pm\sqrt{4+4}}{2}$
	$x=\frac{2\pm\sqrt{8}}{2}$
	$x=\frac{2\pm 2\sqrt{2}}{2}=\frac{2\left(1\pm\sqrt{2}\right)}{2}=1\pm\sqrt{2}$

Sol. set $=\left\{\pm 1,\ 1\pm\sqrt{2}\right\}$

11. $2x^4+x^3-6x^2+x+2=0$

Solution: $2x^4+x^3-6x^2+x+2=0$

Dividing by x^2, we get

$$2x^2+x-6+\frac{1}{x}+\frac{2}{x^2}=0$$

regrouping

$$\left(2x^2+\frac{2}{x^2}\right)+\left(x+\frac{1}{x}\right)-6=0$$

$$2\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-6=0 \qquad (A)$$

Let $x+\frac{1}{x}=y$ (squaring)

$$x^2+\frac{1}{x^2}+2=y^2$$

$$x^2+\frac{1}{x^2}=y^2-2$$ equation A becomes.

$$2\left(y^2-2\right)+y-6=0$$

$$2y^2-4+y-6=0$$

$$2y^2+y-10=0$$

$$2y^2+5y-4y-10=0$$

$$y(2y+5)-2(2y+5)=0$$

$$(2y+5)(y-2)=0$$

$y-2=0$ or $2y+5=0$

$y=2$ | $2y=-5$

$$y=\frac{-5}{2}$$

If $y=2$ | If $y=-\frac{5}{2}$

then $x+\frac{1}{x}=2$ (multiplying by x) | then $x+\frac{1}{x}=-\frac{5}{2}$ (multiplying by $2x$)

$x^2+1=2x$
$x^2-2x+1=0$
$(x-1)^2=0$
$x-1=0$
$x=1$

$2x^2+2=-5x$
$2x^2+5x+2=0$
$2x^2+4x+x+2=0$
$2x(x+2)+1(x+2)=0$
$(x+2)(2x+1)=0$
$2x+1=0$
$2x=-1$
$x+2=0$
gives $x=-\frac{1}{2}$
gives $x=-2$

Sol. set $=\left\{1,-2,-\frac{1}{2}\right\}$

12. $4.2^{2x+1}-9.2^x+1=0$

Solution: $4.2^{2x+1}-9.2^x+1=0$

$4.2^{2x}.2-9.2^x+1=0$
$4.2.2^{2x}-9.2^x+1=0$

Let $2^x=y$, then $2^{2x}=y^2$, the given equation becomes

$4.2.y^2-9y+1=0$
$8y^2-9y+1=0$
$8y^2-y-8y+1=0$
$y(8y-1)-1(8y-1)=0$
$(8y-1)(y-1)=0$
$y-1=0$ or $8y-1=0$
$y=1$

$8y=1$
$y=\frac{1}{8}$

If $y=1$

If $y=\frac{1}{-}$

then $2^x = 1$

$2^x = 2^0$ N.T.S

Thus, $x = 0$

$y = \frac{1}{2^3}$

then $2^x = 2^{-3}$

Thus, $x = -3$

Sol. set $= \{0, -3\}$

13. $3^{2x+2} = 12.3^x - 3$

Solution: $3^{2x+2} = 12.3^x - 3$

$3^{2x}.3^2 = 12.3^x - 3$

$9.3^{2x} = 12.3^x - 3$

Let $3^x = y$, then $3^{2x} = y^2$, the given equation becomes

$9y^2 = 12y - 3$

$9y^2 - 12y + 3 = 0$ (Dividing by 3)

$3y^2 - 4y + 1 = 0$

$3y^2 - 3y - y + 1 = 0$

$3y(y-1) - 1(y-1) = 0$

$(y-1)(3y-1) = 0$

$y - 1 = 0$ or $3y - 1 = 0$

$y = 1$

If $y = 1$

then $3^x = 1$

$3^x = 3^0$ N.T.S

Thus, $x = 0$

$3y = 1$

$y = \frac{1}{3}$

If $y = \frac{1}{3}$

then $3^x = \frac{1}{3}$

$3^x = 3^{-1}$

Thus, $x = -1$

Sol. set $= \{0, -1\}$

14. $2^x + 64.2^{-x} - 20 = 0$

Solution: $2^x + 64.2^{-x} - 20 = 0$

Let $2^x = y$, then $2^{-x} = y^{-1} = \frac{1}{y}$, the given equation becomes

$$y + \frac{64}{y} - 20 = 0 \text{ (multiplying by } y\text{)}$$

$$y^2 + 64 - 20y = 0$$

$$y^2 - 20y + 64 = 0$$

$$y^2 - 4y - 16y + 64 = 0$$

$$y(y-4) - 16(y-4) = 0$$

$$(y-4)(y-16) = 0$$

$y - 4 = 0$ or $y - 16 = 0$

$y = 4$	$y = 16$
If $y = 4$	If $y = 16$
then $2^x = 4$	then $2^x = 16$
$2^x = 2^2$	$2^x = 2^4$
Thus, $x = 2$	Thus, $x = 4$

Sol. set = $\{2, 4\}$

15. $(x+1)(x+3)(x-5)(x-7) = 192$

Solution: $(x+1)(x+3)(x-5)(x-7) = 192$

Regrouping

$$\{(x+1)(x-5)\}\{(x+3)(x-7)\} = 192$$

$$(x^2 - 4x - 5)(x^2 - 4x - 21) = 192$$

Let $x^2 - 4x = y$, the given equation becomes

$$(y-5)(y-21) = 192$$

$$y^2 - 26y + 105 - 192 = 0$$

$$y^2 - 26y - 87 = 0$$

$$y^2 - 29y + 3y - 87 = 0$$

$y(y-29)+3(y-29)=0$

$(y-29)(y+3)=0$

$y-29=0$ or $y+3=0$

$y=29$ | $y=-3$

If $y=29$

then $x^2-4x=29$

$x^2-4x-29=0$

$a=1,\ b=-4,\ c=-29$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-29)}}{2(1)}$$

$$x=\frac{4\pm\sqrt{16+116}}{2}$$

$$x=\frac{4\pm\sqrt{132}}{2}$$

$$x=\frac{4\pm\sqrt{4\times 33}}{2}$$

$$x=\frac{4\pm 2\sqrt{33}}{2}$$

$$x=\frac{2\left(2\pm\sqrt{33}\right)}{2}$$

$x=2\pm\sqrt{33}$

If $y=-3$

then $x^2-4x=-3$

$x^2-4x+3=0$

$x^2-x-3x+3=0$

$x(x-1)-3(x-1)=0$

$(x-1)(x-3)=0$

$x-1=0,\quad x-3=0$

$x=1,\quad x=3$

Sol. set $=\left(1,3,2\pm\sqrt{33}\right)$

16. $(x-1)(x-2)(x-8)(x+5)+360=0$

$$[(x-1)(x-2)][(x-8)(x+5)]+360=0$$
$$(x^2-3x+2)(x^2-3x-40)+360=0 \quad (A)$$

Let $x^2-3x=y$, A becomes

$$(y+2)(y-40)+360=0$$
$$y^2-38y-80+360=0$$
$$y^2-38y+280=0$$
$$y^2-10y-28y+280=0$$
$$y(y-10)-28(y-10)=0$$
$$y-10=0 \quad \text{or} \quad y-28=0$$

$y=10$

If $y=10$

then $x^2-3x=10$

$$x^2-3x-10=0$$
$$x^2-5x+2x-10=0$$
$$x(x-5)+2(x-5)=0$$
$$(x-5)(x+2)=0$$

then $x-5=0$ or $x+2=0$

gives $x=5,\ x=-2$

$y=28$

If $y=28$

then $x^2-3x=28$

$$x^2-3x-28=0$$
$$x^2-7x+4x-28=0$$
$$x(x-7)+4(x-7)=0$$
$$(x-7)(x+4)=0$$
$$x-7=0 \quad \text{or} \quad x+4=0$$

gives $x=7,\ x=-4$

Sol. set $=\{5,-2,7,-4\}$

EXERCISE 1.4

Solve the following equations.

1. $2x+5=\sqrt{7x+16}$

Solution: $2x+5=\sqrt{7x+16}$ (squaring both sides)

$$(2x+5)^2=\left(\sqrt{7x+16}\right)^2$$

$$4x^2+20x+25=7x+16$$

$$4x^2+20x+25-7x-16=0$$

$$4x^2+13x+9=0$$

$$4x^2+4x+9x+9=0$$

$$4x(x+1)+9(x+1)=0$$

$$(x+1)(4x+9)=0$$

$x+1=0$ or $4x+9=0$

$x=-1$ | $4x=-9$

$x=-\frac{9}{4}$

Sol. set $=\left\{-1,-\frac{9}{4}\right\}$

2. $\sqrt{x+3}=3x-1$

Solution: $\sqrt{x+3}=3x-1$ squaring both sides

$$\left(\sqrt{x+3}\right)^2=(3x-1)^2$$

$$x+3=9x^2-6x+1$$

$$9x^2-6x-x+1-3=0$$

$$9x^2-7x-2=0$$

$$9x^2-9x+2x-2=0$$

$$9x(x-1)+2(x-1)=0$$

$(x-1)(9x+2)=0$

$x-1=0$ or $9x+2=0$

$x=1$ | $9x=-2$

$x=-\dfrac{2}{9}$

Sol. set $=\left\{1,-\dfrac{2}{9}\right\}$

3. $4x=\sqrt{13x+14}-3$

Solution: $4x=\sqrt{13x+14}-3$

$4x+3=\sqrt{13x+14}$ (squaring both sides)

$(4x+3)^2=\left(\sqrt{13x+14}\right)^2$

$16x^2+24x+9-13x-14=0$

$16x^2+11x-5=0$

$16x^2+16x-5x-5=0$

$16x(x+1)-5(x+1)=0$

$(x+1)(16x-5)=0$

$x+1=0$ or $16x-5=0$

$x=-1$ | $16x=5$

$x=\dfrac{5}{16}$

Sol. set $=\left(-1,\ \dfrac{5}{16}\right)$

4. $\sqrt{3x+100}-x=4$

Solution: $\sqrt{3x+100}-x=4$

$\sqrt{3x+100}=4+x$

$\Rightarrow$ $4+x=\sqrt{3x+100}$ (squaring both sides)

$$(4+x)^2 = \left(\sqrt{3x+100}\right)^2$$
$$16+8x+x^2 = 3x+100$$
$$x^2+8x+16-3x-100=0$$
$$x^2+5x-84=0$$
$$x^2+12x-7x-84=0$$
$$x(x+12)-7(x+12)=0$$
$$(x+12)(x-7)=0$$
$x-7=0$ or $x+12=0$

$x=7$, $x=-12$

Sol. set = { 7, −12}

5. $\sqrt{x+5}+\sqrt{x+21}=\sqrt{x+60}$

Solution: $\sqrt{x+5}+\sqrt{x+21}=\sqrt{x+60}$ (squaring both sides)

$$\left(\sqrt{x+5}+\sqrt{x+21}\right)^2 = \left(\sqrt{x+60}\right)^2$$
$$x+5+x+21+2\sqrt{(x+5)(x+21)} = x+60$$
$$2x+26+2\sqrt{x^2+26x+105} = x+60$$
$$2\sqrt{x^2+26x+105} = x+60-2x-26$$
$2\sqrt{x^2+26x+105} = 34-x$ (squaring again)
$$4(x^2+26x+105) = 1156-68x+x^2$$
$$4x^2+104x+420-x^2+68x-1156=0$$
$$3x^2+172x-736=0$$
$$3x^2-12x+184x-736=0$$
$$3x(x-4)+184(x-4)=0$$
$$(x-4)(3x+184)=0$$
$x-4=0$ or $3x+184=0$

$x=4$ | $3x=-184$

$$x = -\frac{184}{3}$$

Sol. set $= \left\{4, -\frac{184}{3}\right\}$

6. $\sqrt{x+1} + \sqrt{x-2} = \sqrt{x+6}$

Solution: $\sqrt{x+1} + \sqrt{x-2} = \sqrt{x+6}$ (squaring both sides)

$$\left(\sqrt{x+1} + \sqrt{x-2}\right)^2 = \left(\sqrt{x+6}\right)^2$$
$$(x+1) + (x-2) + 2\sqrt{(x+1)(x-2)} = x+6$$
$$x+1+x-2+2\sqrt{x^2-x-2} = x+6$$
$$2x-1+2\sqrt{x^2-x-2} = x+6$$
$$2\sqrt{x^2-x-2} = x+6-2x+1$$
$$2\sqrt{x^2-x-2} = 7-x \quad \text{(squaring again)}$$
$$4(x^2-x-2) = 49-14x+x^2$$
$$4x^2-4x-8 = 49-14x+x^2$$
$$4x^2-x^2-4x+14x-8-49 = 0$$
$$3x^2+10x-57 = 0$$
$$3x^2-9x+19x-57 = 0$$
$$3x(x-3)+19(x-3) = 0$$
$$(x-3)(3x+19) = 0$$

$x-3=0$ or $3x+19=0$

Thus $x=3$

$$3x = -19$$
$$x = -\frac{19}{3}$$

Sol. set $= \left\{3, -\frac{19}{3}\right\}$

7. $\sqrt{11-x}-\sqrt{6-x}=\sqrt{27-x}$

Solution: $\sqrt{11-x}-\sqrt{6-x}=\sqrt{27-x}$ (squaring both sides)

$$\left(\sqrt{11-x}-\sqrt{6-x}\right)^2=\left(\sqrt{27-x}\right)^2$$

$$(11-x)+(6-x)-2\sqrt{(11-x)(6-x)}=27-x$$

$$11-x+6-x-2\sqrt{66-17x+x^2}=27-x$$

$$17-2x-2\sqrt{66-17x+x^2}=27-x$$

$$-2\sqrt{66-17x+x^2}=27-x-17+2x$$

$$-2\sqrt{66-17x+x^2}=10+x \quad \text{(squaring again)}$$

$$4(66-17x+x^2)=100+20x+x^2$$

$$264-68x+4x^2=100+20x+x^2$$

$$4x^2-x^2-68x-20x+264-100=0$$

$$3x^2-88x+164=0$$

$$3x^2-6x-82x+164=0$$

$$3x(x-2)-82(x-2)=0$$

$$(x-2)(3x-82)=0$$

$x-2=0$ or $3x-82=0$

$x=2$ | $3x=82$

$x=\dfrac{82}{3}$

Sol. set $=\left\{2,\ \dfrac{82}{3}\right\}$

8. $\sqrt{4a+x}-\sqrt{a-x}=\sqrt{a}$

Solution: $\sqrt{4a+x}-\sqrt{a-x}=\sqrt{a}$ (squaring both sides)

$$\left(\sqrt{4a+x}-\sqrt{a-x}\right)^2=\left(\sqrt{a}\right)^2$$

$$(4a+x)+(a-x)-2\sqrt{(4a+x)(a-x)}=a$$

$$4a+x+a-x-2\sqrt{(4a+x)(a-x)}=a$$

$$5a-2\sqrt{(4a+x)(a-x)}=a$$

$$-2\sqrt{(4a+x)(a-x)}=a-5a$$

$$-2\sqrt{(4a+x)(a-x)}=a-4a$$

$$\sqrt{(4a+x)(a-x)}\frac{-4a}{-2}=2a$$

$$\left(\sqrt{(4a+x)(a-x)}\right)^2=(2a)^2 \quad \text{(squaring again)}$$

$$(4a+x)(a-x)=4a^2$$

$$\cancel{4a^2}-4ax+ax-x^2=\cancel{4a^2}$$

$$-x^2-3ax=0$$

$$-x(x+3a)=0$$

$$-x=0, \quad x+3a=0$$

$$x=0 \quad x=-3a$$

Solutionset set $=\{0\},(-3a, \textit{extraneous root})$

9. $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$

Solution: $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$

$$\sqrt{x^2+x+1}=1+\sqrt{x^2+x-1} \quad \text{(squaring both sides)}$$
$$x^2+x+1=(1)^2+(x^2+x-1)+2(1)\sqrt{x^2+x-1}$$
$$x^2+x+1=1+x^2+x-1+2\sqrt{x^2+x-1}$$
$$x^2+x+1-x^2-x=2\sqrt{x^2+x-1}$$
$$1=2\sqrt{x^2+x-1} \quad \text{(squaring again)}$$
$$1=4(x^2+x-1)$$
$$1=4x^2+4x-4$$
$$0=4x^2+4x-4-1$$
$$0=4x^2+4x-5$$

or $4x^2+4x-5=0$

Here, $a=4, b=4, c=-5$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-4\pm\sqrt{(4)^2-4(4)(-5)}}{2(4)}$$
$$x=\frac{-4\pm\sqrt{16+80}}{8}$$
$$x=\frac{-4\pm\sqrt{96}}{8}$$
$$x=\frac{-4\pm\sqrt{4\times4\times6}}{8}$$
$$x=\frac{-4\pm4\sqrt{6}}{8}$$
$$x=\frac{4\left(-1\pm\sqrt{6}\right)}{8}$$

$$x = \frac{-1 \pm \sqrt{6}}{2}$$

$$\text{Sol. set} = \left\{\frac{-1 \pm \sqrt{6}}{2}\right\}$$

10. $\sqrt{x^2 + 3x + 8} + \sqrt{x^2 + 3x + 2} = 3$

Solution: $\sqrt{x^2 + 3x + 8} + \sqrt{x^2 + 3x + 2} = 3$

$$\sqrt{x^2 + 3x + 8} = 3 - \sqrt{x^2 + 3x + 2} \quad \text{(squaring both sides)}$$

$$x^2 + 3x + 8 = 9 + (x^2 + 3x + 2) - 6\sqrt{x^2 + 3x + 2}$$

$$x^2 + 3x + 8 = 9 + x^2 + 3x + 2 - 6\sqrt{x^2 + 3x + 2}$$

$$x^2 + 3x + 8 - 9 - x^2 - 3x - 2 = -6\sqrt{x^2 + 3x + 2}$$

$$-3 = -6\sqrt{x^2 + 3x + 2}$$

$$-1 = -2\sqrt{x^2 + 3x + 2} \quad \text{(squaring again)}$$

$$1 = 4(x^2 + 3x + 2)$$

$$1 = 4x^2 + 12x + 8$$

$$0 = 4x^2 + 12x + 8 - 1$$

$$0 = 4x^2 + 12x + 7$$

or $4x^2 + 12x + 7 = 0$

Here, $a = 4, b = 12, c = 7$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4(4)(7)}}{2(4)}$$

$$x = \frac{-12 \pm \sqrt{144 - 112}}{8}$$

$$x=\frac{-12\pm\sqrt{32}}{8}$$
$$x=\frac{-12\pm\sqrt{4\times4\times2}}{8}$$
$$x=\frac{-12\pm4\sqrt{2}}{8}$$
$$x=\frac{4\left(-3\pm\sqrt{2}\right)}{8}$$
$$x=\frac{-3\pm\sqrt{2}}{2}$$

Solution set $=\left\{\frac{-3\pm\sqrt{2}}{2}\right\}$

11. $\sqrt{x^2+3x+9}+\sqrt{x^2+3x+4}=5$

Solution: $\sqrt{x^2+3x+9}+\sqrt{x^2+3x+4}=5$

$\sqrt{x^2+3x+9}=5-\sqrt{x^2+3x+4}$ (squaring, both sides)

$x^2+3x+9=25+x^2+3x+4-10\sqrt{x^2+3x+4}$

$x^2+3x+9-25-x^2-3x-4=-10\sqrt{x^2+3x+4}$

$-20=-10\sqrt{x^2+3x+4}$ (Dividing by -10)

$2=\sqrt{x^2+3x+4}$ (squaring again)

$4=x^2+3x+4$

$0=x^2+3x+4-4$

$0=x^2+3x$

or $x^2+3x=0$

$x(x+3)=0$

$x=0$ or $x+3=0$

$x=-3$

Sol. set $=\{0,-3\}$

EXERCISE 2.1

1. Find the discriminant of the following given quadratic equations:

(i) $2x^2 + 3x - 1 = 0$

Solution: $2x^2 + 3x - 1 = 0$

Here, $a = 2,\ b = 3,\ c = -1$

$$\begin{aligned} \text{Disc.} &= b^2 - 4ac \\ &= (3)^2 - 4(2)(-1) \\ &= 9 + 8 = 17 \text{ Ans.} \end{aligned}$$

(ii) $6x^2 - 8x + 3 = 0$

Solution: $6x^2 - 8x + 3 = 0$

Here, $a = 6,\ b = -8,\ c = 3$

$$\begin{aligned} \text{Disc.} &= b^2 - 4ac \\ &= (-8)^2 - 4(6)(3) \\ &= 64 - 72 \\ &= -8 \text{ Ans.} \end{aligned}$$

(iii) $9x^2 - 30x + 25 = 0$

Solution: $9x^2 - 30x + 25 = 0$

Here, $a = 9,\ b = -30,\ c = 25$

$$\begin{aligned} \text{Disc.} &= b^2 - 4ac \\ &= (-30)^2 - 4(9)(25) \\ &= 900 - 900 \\ &= 0 \text{ Ans.} \end{aligned}$$

(iv) $4x^2 - 7x - 2 = 0$

Solution: $4x^2 - 7x - 2 = 0$

Here, $a = 4,\ b = -7,\ c = -2$

$$\begin{aligned} \text{Disc.} &= b^2 - 4ac \\ &= (-7)^2 - 4(4)(-2) \\ &= 49 + 32 \end{aligned}$$

81 **Ans.**

2. ***Find the nature of the roots of the following given quadratic equations and verify the result by solving the equations:***

(i) $x^2 - 23x + 120 = 0$

Solution:

$x^2 - 23x + 120 = 0$

Here, $a = 1$, $b = -23$, $c = 120$

$$\text{Disc.} = b^2 - 4ac$$
$$= (-23)^2 - 4(1)(120)$$
$$= 529 - 480$$
$$= 49 \text{ (perfect square)}$$

The roots are real rational and unequal.

Verification:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-23) \pm \sqrt{(-23)^2 - 4(1)(120)}}{2(1)}$$
$$x = \frac{23 \pm \sqrt{529 - 480}}{2}$$
$$x = \frac{23 \pm \sqrt{49}}{2}$$
$$x = \frac{23 \pm 7}{2}$$

Either $x = \frac{23 + 7}{2}$, $x = \frac{23 - 7}{2}$

$x = \frac{30}{2}$, $x = \frac{16}{2}$

$x = 15$, $x = 8$

The roots are real, rational and unequal,

(ii) $2x^2 + 3x + 7 = 0$

Solution:

Here, $a = 2, b = 3, c = 7$

$$\begin{aligned}\text{Disc.} &= b^2 - 4ac \\ &= (3)^2 - 4(2)(7) \\ &= 9 - 56 \\ &= -47\end{aligned}$$

Disc. is negative, therefore, the roots are imaginary and unequal

Verification:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(7)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{9 - 56}}{4}$$

$$x = \frac{-3 \pm \sqrt{-47}}{4}$$

The roots are imaginary and unequal.

(iii) $16x^2 - 24x + 9 = 0$

Solution:

Here, $a = 16, b = -24, c = 9$

$$\begin{aligned}\text{Disc.} &= b^2 - 4ac \\ &= (-24)^2 - 4(16)(9) \\ &= 576 - 576 \\ &= 0 \quad \text{(The roots are real, rational and equal)}\end{aligned}$$

Verification:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(16)(9)}}{2(16)}$$

$$x = \frac{24 \pm \sqrt{576 - 576}}{32}$$

$$x = \frac{24 \pm 0}{32}$$

$$x = \frac{24 + 0}{32} \quad \text{or} \quad x = \frac{24 - 0}{32}$$

$$x = \frac{3}{4} \text{ or } \frac{3}{4}$$ (The roots are real, rational and equal.)

(iv) $3x^2 + 7x - 13 = 0$

Solution: $3x^2 + 7x - 13 = 0$

Here, $a = 3, b = 7, c = -13$

Disc. $= b^2 - 4ac$

$= (7)^2 - 4(3)(-13)$

$= 49 + 156$

$= 205$ not a perfect square

The roots are real, irrational and unequal.

Verification:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-13)}}{2(3)}$$

$$x = \frac{-7 \pm \sqrt{49 + 156}}{6}$$

$$x = \frac{-7 \pm \sqrt{205}}{6}$$

So, the roots are real, irrational and unequal.

3. For what value of k, the expression $k^2x^2+2(k+1)x+4$ is perfect square.

Solution: Let $k^2x^2 + 2(k+1)x + 4 = 0$, left side will be a perfect square if its Disc.= is zero.

Here, $a = k^2$, $b = 2(k+1)$, $c = 4$

Disc.$= b^2 - 4ac$

Putting values of a, b, c, we get

$$= [2(k+1)]^2 - 4(k^2)(4)$$
$$= 4(k^2 + 2k + 1) - 16k^2$$
$$= 4k^2 + 8k + 4 - 16k^2$$
$$8k + 4 - 12k^2 = 0 \quad \text{(according to the condition)}$$
$$12k^2 - 8k - 4 = 0 \quad \text{(Dividing by 4)}$$
$$3k^2 - 2k - 1 = 0$$
$$3k^2 - 3k + k - 1 = 0$$
$$3k(k-1) + 1(k-1) = 0$$
$$(k-1)(3k+1) = 0$$
$$k - 1 = 0 \quad \text{or} \quad 3k + 1 = 0$$
$$3k = -1$$

Thus, $k = 1$ $\quad$ $k = -\frac{1}{3}$

4. Find the value of k, if the roots of the following equations are equal.

(i) $(2k-1)x^2 + 3kx + 3 = 0$

Solution: $(2k-1)x^2 + 3kx + 3 = 0$

Here, $a = 2k - 1$, $b = 3k$, $c = 3$

For equal roots Disc. must be zero.

Disc.$= b^2 - 4ac = 0$

Putting value of a, b, c, we get

$$(3k)^2 - 4(2k-1)(3) = 0$$
$$9k^2 - 24k + 12 = 0 \quad \text{Dividing by 3}$$
$$3k^2 - 8k + 4 = 0$$
$$3k^2 - 2k - 6k + 4 = 0$$

$$k(3k-2)-2(3k-2)=0$$
$$(k-2)(3k-2)=0$$
$$k-2=0 \quad \text{or} \quad 3k-2=0$$
$$3k=2$$
$$\text{Thus,} \quad k=2 \qquad k=\frac{2}{3}$$

(ii) $\boldsymbol{x^2+2(k+2)x+(3k+4)=0}$

Solution: $\boldsymbol{x^2+2(k+2)x+(3k+4)=0}$

Here, $a=1$, $b=2(k+2)$, $c=3k+4$

Disc. $=b^2-4ac=0$

Putting values of a, b, c, we get

$$[2(k+2)]^2-4(1)(3k+4)=0$$
$$4(k^2+4k+4)-12k-16=0$$
$$4k^2+16k+16-12k-16=0$$
$$4k^2+4k=0$$
$$4k(k+1)=0$$
$$4k=0 \quad \text{or} \quad k+1=0$$
$$k=0 \quad \text{or} \quad k=-1$$

(iii) $\boldsymbol{(3k+2)x^2-5(k+1)x+(2k+3)=0}$

Solution: $\boldsymbol{(3k+2)x^2-5(k+1)x+(2k+3)=0}$

Here, $a=3k+2$, $b=-5(k+1)$, $c=2k+3$

$\because$ Disc. $=b^2-4ac=0$

$\therefore$ Putting values of a, b, c, we get

$$[-5(k+1)]^2-4(3k+2)(2k+3)=0$$
$$25(k^2+2k+1)-4(6k^2+13k+6)=0$$
$$25k^2+50k+25-24k^2-52k-24=0$$
$$k^2-2k+1=0$$
$$(k-1)^2=0$$
$$k-1=0$$
$$k=1$$

5. *Show that the equation $x^2 + (mx + c)^2 = a^2$ has equal roots, if $c^2 = a^2(1 + m^2)$*

Solution: $x^2 + (mx + c)^2 = a^2$

$x^2 + m^2x^2 + 2mcx + c^2 = a^2$

$x^2 + m^2x^2 + 2mcx + c^2 - a^2 = 0$

$(1 + m^2)x^2 + 2mcx + (c^2 - a^2) = 0$

Disc. must be zero for equal roots

Here, $a' = 1 + m^2$, $b' = 2mc$, $c' = c^2 - a^2$

Disc: $(b')^2 - 4a'c' = 0$

Putting values of a', b', c', we get

$(2mc)^2 - 4(1 + m^2)(c^2 - a^2) = 0$

$4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2) = 0$

$4m^2c^2 - 4c^2 + 4a^2 - 4m^2c^2 + 4m^2a^2 = 0$

$-4c^2 = -4m^2a^2 - 4a^2$ (Dividing by -4)

$c^2 = m^2a^2 + a^2$

$c^2 = a^2(m^2 + 1)$

$c^2 = a^2(1 + m^2)$ Hence the result.

6. *Find the condition that the roots of the equation $(mx + c)^2 - 4ax = 0$ are equal.*

Solution: $(mx + c)^2 - 4ax = 0$

$m^2x^2 + 2cmx + c^2 - 4ax = 0$

$m^2x^2 + 2cmx - 4ax + c^2 = 0$

$m^2x^2 + (2cm - 4a)x + c^2 = 0$

The roots will be equal when its Disc. = is zero

Thus, $(b')^2 - 4a'c' = 0$

Here, $b' = 2cm - 4a$, $a' = m^2$, $c' = c^2$

Now $(2cm - 4a)^2 - 4(m^2)(c^2) = 0$

$4c^2m^2 + 16a^2 - 16cma - 4c^2m^2 = 0$

$16a^2 - 16cma = 0$

$16a(a - cm) = 0$

$\Rightarrow a - cm = 0$ as $a \neq 0$

Thus a cm is the required condition.

7. ***If the roots of the equation***
$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal, then $a = 0$ or $a^3 + b^3 + c^3 = 3abc$

Solution:

$$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$$

Disc. is zero, if roots are equal

Therefore;

$$[-2(a^2 - bc)]^2 - 4(c^2 - ab)(b^2 - ac) = 0 \qquad \text{(Disc).}$$
$$4(a^4 - 2a^2bc + b^2c^2) - 4(b^2c^2 - ac^3 - ab^3 + a^2bc) = 0$$

(Dividing by 4, we have)

$$a^4 - 2a^2bc + b^2c^2 - b^2c^2 + ac^3 + ab^3 - a^2bc = 0$$
$$a^4 - 3a^2bc + ac^3 + ab^3 = 0$$
$$a(a^3 - 3abc + c^3 + b^3) = 0$$

Now, $a = 0$ or $a^3 - 3abc + c^3 + b^3 = 0$

Thus, $a^3 - 3abc + c^3 + b^3 = 0$
$$a^3 + b^3 + c^3 = 3abc$$

8. ***Show that the roots of the following equations are rational.***

(i) $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$

Solution:

$$a(b - c)x^2 + b(c - a)x + c(a - b) = 0$$

The roots will be rational, if Disc. is a perfect square.

Disc: $[b(c - a)]^2 - 4a(b - c)(c)(a - b)$
$$b^2(c^2 + a^2 - 2ac) - 4ac(ab - b^2 - ac + bc)$$
$$b^2c^2 + a^2b^2 - 2ab^2c - 4a^2bc + 4ab^2c + 4a^2c^2 - 4abc^2$$
$$b^2c^2 + 2ab^2c + a^2b^2 - 4a^2bc - 4abc^2 + 4a^2c^2$$
$$b^2c^2 + 2ab^2c + a^2b^2 - 4abc(a + c) + 4a^2c^2$$
$$b^2(c^2 + 2ac + a^2) - 4abc(a + c) + (2ac)^2$$
$$b^2(c + a)^2 - 2\{b(a + c)(2ac)\} + (2ac)^2$$
$$\{b(c + a)\}^2 - 2\{b(c + a)(2ac)\} + (2ac)^2$$
$$\{b(c + a) - 2ac\}^2 \quad \text{which is a perfect square.}$$

Thus, the roots are rational.

(ii) $(a + 2b)x^2 + 2(a + b + c)x + (a + 2c) = 0$

Solution:

$(a + 2b)x^2 + 2(a + b + c)x + (a + 2c) = 0$

The roots will be rational, if Disc. – is a perfect square.

Disc: $[2(a + b + c)]^2 - 4(a + 2b)(a + 2c)$

$= 4(a + b + c)^2 - 4(a^2 + 2ca + 2ab + 4bc)$

$= 4(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca) - 4(a^2 + 2ca + 2ab + 4bc)$

$= 4[a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - a^2 - 2ca - 2ab - 4bc]$

$= 4[b^2 + c^2 - 2bc]$

$= 4(b - c)^2$

$= [2(b - c)]^2$ which is a perfect square.

Hence, the roots are rational.

9. ***For all values of k, prove that the roots of the equation. $x^2 - 2\left(k + \frac{1}{k}\right)x + 4 = 0,\ k \neq 0$ are real.***

Solution:

$x^2 - 2\left(k + \frac{1}{k}\right)x + 4 = 0$

For real roots Disc. must be positive.

Here, $a = 1,\ b = -2\left(k + \frac{1}{k}\right),\ c = 4$

Disc: $b^2 - 4ac$

$= \left[-2\left(k + \frac{1}{k}\right)\right]^2 - 4(1)(4)$

$= 4\left(k + \frac{1}{k}\right)^2 - 16$

$= 4\left(k^2 + \frac{1}{k^2} + 2\right) - 16$

$= 4k^2 + \frac{4}{k^2} + 8 - 16$

$4k^2 + \frac{4}{k^2} - 8$

$4\left[k^2 + \frac{1}{k^2} - 2\right]$

$4\left[k - \frac{1}{k}\right]^2$

$\left[2\left(k - \frac{1}{k}\right)\right]^2$ which is positive for all values of x.

10. Show that the roots of the equation
$(b-c)x^2 + (c-a)x + (a-b) = 0$ are real.

Solution:

$(b-c)x^2 + (c-a)x + (a-b) = 0$

For real roots Disc. must be positive.

Disc: $(b')^2 - 4a'c'$

$(c-a)^2 - 4(b-c)(a-b)$

$c^2 + a^2 - 2ca - 4(ab - b^2 - ca + bc)$

$c^2 + a^2 - 2ca - 4ab + 4b^2 + 4ca - 4bc$

$c^2 + a^2 + 4b^2 + 2ca - 4bc - 4ab$

$\left[(c)^2 + (a)^2 + (2b)^2 + 2(c)(a) - 2(a)(2b) - 2(c)(2b)\right]$

$(c + a - 2b)^2$ which is always positive.

Hence, the roots are real.

EXERCISE 2.2

1. ***Find the cube roots of −1, 8, − 27, 64.***

Solution: Cube root of −1.

Let $x^3 = -1$

$$x^3 + 1 = 0$$

$$x^3 + 1^3 = 0$$

$$(x+1)(x^2 - x + 1) = 0$$

Then $x + 1 = 0$ gives

$$x = -1$$

and $x^2 - x + 1 = 0$

Here, $a = 1, b = -1, c = 1$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1-4}}{2}$$

$$x = \frac{1 \pm \sqrt{-3}}{2}$$

Thus, $x = -1, \frac{1 + \sqrt{-3}}{2}, \frac{1 - \sqrt{-3}}{2}$

$$x = -1, \frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}$$

$$x = -1, -\omega, -\omega^2$$

(ii) Cube root of 8

Let $x^3 = 8$

$$x^3 - 8 = 0$$

$$x^3 - 2^3 = 0$$

$$(x-2)(x^2+2x+4) = 0$$

Therefore, $x - 2 = 0$

$$x = 2$$

and $x^2 + 2x + 4 = 0$

Here, $a = 1,\ b = 2,\ c = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4 - 16}}{2}$$

$$x = \frac{-2 \pm \sqrt{-12}}{2}$$

$$x = \frac{-2 \pm 2\sqrt{-3}}{2}$$

$$x = \frac{2\left(-1 \pm \sqrt{-3}\right)}{2}$$

$$x = -1 \pm \sqrt{-3}$$

$$= \left(-1 \pm i\sqrt{3}\right)\left(\frac{2}{2}\right) \text{ N.T.S.}$$

$$= \frac{2\left(-1 \pm i\sqrt{3}\right)}{2}$$

$$= 2\omega,\ 2\omega^2$$

Roots are: $2,\ 2\omega,\ 2\omega^2$

(iii) ***Cube root of – 27***

Let $x^3 = -27$

$$x^3 + 27 = 0$$

$$x^3 + 3^3 = 0$$

$(x + 3)(x^2 - 3x + 9) = 0$ which gives .

$x + 3 = 0$ i.e., $x = -3$

and $x^2 - 3x + 9 = 0$

Here, $a = 1, b = -3, c = 9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{9 - 36}}{2}$$

$$x = \frac{3 \pm \sqrt{-27}}{2}$$

$$x = \frac{3 \pm 3\sqrt{-3}}{2}$$

$$x = \frac{3\left(1 \pm i\sqrt{3}\right)}{2}$$

$$x = -3\omega, -3\omega^2$$

Roots are : $-3, -3\omega, -3\omega^2$

(iv) ***Cube root of 64.***

Let $x^3 = 64$

$$x^3 = 4^3$$

$$x^3 - 4^3 = 0$$

$(x - 4)(x^2 + 4x + 16) = 0$ this gives

$$x - 4 = 0$$

$$x = 4$$

and $x^2 + 4x + 16 = 0$

Here, $a = 1, b = 4, c = 16$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(16)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 - 64}}{2}$$

$$x = \frac{-4 \pm \sqrt{-48}}{2}$$

$$x = \frac{-4 \pm 4\sqrt{-3}}{2}$$

$$x = \frac{4\left(-1 \pm i\sqrt{3}\right)}{2}$$

$$x = 4\omega, 4\omega^2$$

Roots are : $4, 4\omega, 4\omega^2$

2. *Evaluate*

(i) $(1 - \omega - \omega^2)^7$

Solution: $(1 - \omega - \omega^2)^7$

$= [(1 - (\omega + \omega^2)]^7$ $\left[\begin{array}{r} \text{using } 1 + \omega + \omega^2 = 0 \\ \omega + \omega^2 = -1 \end{array}\right]$

$= [(1 - (-1)]^7$

$= (1 + 1)^7$

$= (2)^7$

$= 128$ **Ans.**

(ii) $(1 - 3\omega - 3\omega^2)^5$

Solution: $(1 - 3\omega - 3\omega^2)^5$

$= [1 - 3(\omega + \omega^2)]^5$

$= [1 - 3(-1)]^5$ $[\omega + \omega^2 = -1]$

$= (1+3)^5$

$= (4)^5$

$=$ 1024 **Ans.**

(iii) $(9+4\omega+4\omega^2)^3$

Solution: $(9+4\omega+4\omega^2)^3$

$= [(9+4(\omega+\omega^2)]^3$

$= [9+4(-1)]^3$ $\quad$ [as $\omega+\omega^2=-1$]

$= (9-4)^3$

$= (5)^3 = 125$ **Ans.**

(iv) $(2+2\omega-2\omega^2)(3-3\omega+3\omega^2)$

Solution: $(2+2\omega-2\omega^2)(3-3\omega+3\omega^2)$

$= [2(1+\omega)-2\omega^2)][3+3\omega^2-3\omega]$

$= [2(-\omega^2)-2\omega^2)][3(1+\omega^2)-3\omega]$

$= [-2\omega^2-2\omega^2][3(-\omega)-3\omega]$

$= [-4\omega^2][-3\omega-3\omega]$

$= [-4\omega^2][-6\omega]$

$= 24\omega^3$

$= 24\times 1$ $\quad$ $[\omega^3=1]$

$=$ 24 **Ans.**

(v) $(-1+\sqrt{-3})^6+(-1-\sqrt{-3})^6$

Solution: $(-1+\sqrt{-3})^6+(-1-\sqrt{-3})^6$

$$= \left[\frac{2\left(-1+\sqrt{-3}\right)}{2}\right]^6+\left[\frac{2\left(-1-\sqrt{-3}\right)}{2}\right]^6$$

$= (2\omega)^6+(2\omega^2)^6$

$= 64\,\omega^6+64\omega^{12}$

$= 64[\omega^6+\omega^{12}]$

$= 64\,[(\omega^3)^2+(\omega^3)^4]$

$= 64\,[1^2+1^4]$

$64(1+1)$

64×2

128 **Ans.**

(vi) $\left(\frac{-1+\sqrt{-3}}{2}\right)^9 + \left(\frac{-1-\sqrt{-3}}{2}\right)^9$

Solution: $\left(\frac{-1+\sqrt{-3}}{2}\right)^9 + \left(\frac{-1-\sqrt{-3}}{2}\right)^9$

$= \omega^9 + (\omega^2)^9$

$= \omega^9 + \omega^{18}$

$= (\omega^3)^3 + (\omega^3)^6$

$= 1^3 + 1^6$

$= 1 + 1 = 2$ **Ans.**

(vii) $\omega^{37} + \omega^{38} - 5$

Solution: $\omega^{37} + \omega^{38} - 5$

$\omega.\omega^{36} + \omega^2\omega^{36} - 5$

$\omega(\omega^3)^{12} + \omega^2(\omega^3)^{12} - 5$

$\omega(1)^{12} + \omega^2(1)^{12} - 5$

$= \omega + \omega^2 - 5$

$-1 - 5$ $[\omega + \omega^2 = -1]$

-6 **Ans.**

(viii) $\omega^{13} + \omega^{17}$

Solution: $\omega^{13} + \omega^{17}$

$\omega^{12}\omega^{-1} + \omega^{-15}\omega^{-2}$

$(\omega^3)^{-4}\omega^{-1} + (\omega^3)^{-5}\omega^{-2}$

$= (1)^{-4}\omega^{-1} + (1)^{-5}\omega^{-2}$

$= \omega^{-1} + \omega^{-2}$

$\frac{1}{\omega} + \frac{1}{\omega^2}$

$\dfrac{\omega^2+\omega}{\omega^3}$ N.T.S.

$\dfrac{-1}{1}$

-1 **Ans.**

3. Prove that $x^3+y^3=(x+y)(x+\omega y)(x+\omega^2 y)$

Solution: $x^3+y^3=(x+y)(x+\omega y)(x+\omega^2 y)$

R.H.S $(x+y)(x+\omega y)(x+\omega^2 y)$

$(x+y)[x^2+\omega^2 xy+\omega xy+\omega^3 y^2]$

$(x+y)[x^2+(\omega^2+\omega)xy+\omega^3 y^2]$

$(x+y)[x^2+(-1)xy+(1)y^2]$

$(x+y)(x^2-xy+y^2)$

$x^3+y^3=$ L.H.S.

4. Prove that

$x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$.

Solution: $x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

R.H.S $(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

$(x+y+z)[x^2+\omega^2 xy+\omega xz+\omega xy+\omega^3 y^2+\omega^2 yz+$
$\omega^2 xz+\omega^4 yz+\omega^3 z^2]$

$(x+y+z)[x^2+(\omega^2+\omega)xy+(\omega+\omega^2)(xz)+\omega^3 z^2$
$+\omega^3 y^2+(\omega^2+\omega^4)yz]$

$(x+y+z)[x^2+(-1)xy+(-1)xz+(1)z^2+(1)y^2+\omega^2(1+\omega^2)yz]$

$(x+y+z)[x^2-xy-xz+z^2+y^2+\omega^2(-\omega)yz]$

$(x+y+z)[x^2-xy-xz+z^2+y^2-\omega^3 yz]$

$(x+y+z)[x^2-xy-xz+z^2+y^2+\omega^2(-\omega)yz]$

$(x+y+z)(x^2-xy-xz+z^2+y^2-yz)$

$x^3-x^2y-x^2z+xz^2+xy^2-xyz+x^2y-xy^2-xyz+yz^2$
$+y^3-y^2z+x^2z-xyz-xz^2+z^3+y^2z-yz^2$

$x^3+y^3+z^3-3xyz=$ L.H.S.

5. Prove that **$(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots 2n$ factors = 1.**

Solution: $\{(1+\omega)(1+\omega^2)\}\{(1+\omega.\omega^3)(1+\omega^6.\omega^2)\}\ldots$ $2n$ factors

$\{(1+\omega)(1+\omega^2)\}\{(1+\omega)(1+\omega^2)\}\ldots\ldots$ n factors

(as $\omega^3=\omega^6=1$)

$\{(1+\omega)(1+\omega^2)\}^n$

$\{1+\omega^2+\omega+\omega^3\}^n$

$\{(1+\omega+\omega^2)+\omega^3\}^n$

$\{0+1\}^n$ as $1+\omega+\omega^2=0$ and $\omega^3=1$

$(1)^n = 1$ proved.

Roots and Coefficients of a quadratic equation

Let $ax^2+bx+c=0$ be the standard quadratic equation, its roots are:

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a},\ \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Let these roots be α, β, then:

Sum of the roots $= \alpha+\beta = -\frac{b}{a}$

Product of the roots $= \alpha\beta = \frac{c}{a}$

$$S = -\frac{b}{a} = -\frac{\text{Co-efficient of } x}{\text{Co-efficient of } x^2}$$

$$P = \frac{c}{a} = \frac{\text{Constant term}}{\text{Co-efficient of } x^2}$$

EXERCISE 2.3

1. ***Without solving, find the sum and the product of the following quadratic equations.***

(i) $x^2 - 5x + 3 = 0$

Here, $a = 1$, $b = -5$, $c = 3$

Let $\propto$, β be its roots, then:

$$S = \propto + \beta = -\frac{b}{a} = -\frac{(-5)}{1} = 5$$

$$P = \propto\beta = \frac{c}{a} = \frac{3}{1} = 3$$

(ii) $3x^2 + 7x - 11 = 0$

Here, $a = 3$, $b = 7$, $c = -11$

Let $\propto$, β be its roots, then:

$$S = \propto + \beta = -\frac{b}{a} = -\frac{7}{3}$$

$$P = \propto\beta = \frac{c}{a} = \frac{-11}{3} = -\frac{11}{3}$$

(iii) $px^2 - qx + r = 0$

Here, $a = p$, $b = -q$, $c = r$

Let $\propto$, β be its roots, then:

$$S = \propto + \beta = -\frac{b}{a} = -\frac{(-q)}{p} = \frac{q}{p}$$

$$P = \propto\beta = \frac{c}{a} = \frac{r}{p}$$

(iv) $(a + b)x^2 - ax + b = 0$

Here, $a' = a + b$, $b' = -a$, $c' = b$.

Let $\propto$, β be its roots, then:

$$S = \propto + \beta = -\frac{b'}{a'} = -\frac{(-a)}{(a + b)} = \frac{a}{a + b}$$

$P = \propto\beta = \frac{c'}{a'} = \frac{b}{a+b}$

(v) $(l+m)x^2 + (m+n)x + n - l = 0$

Here, $a = l+m, \quad b = m+n, \quad c = n-l$

Let $\propto, \beta$ be its roots, then:

$S = \propto + \beta = -\frac{b}{a} = -\frac{m+n}{l+m}$

$P = \propto\beta = \frac{c}{a} = \frac{n-l}{l+m}$

(vi) $7x^2 - 5mx + 9n = 0$

Here, $a = 7, \quad b = -5m, \quad c = 9$

Let $\propto, \beta$ be its roots, then:

$S = \propto + \beta = -\frac{b}{a} = -\frac{(-5m)}{7} = \frac{5m}{7}$

$P = \propto\beta = \frac{c}{a} = \frac{9}{7}$

2. ***Find the value of k, if***

(i) ***sum of the roots of the equation $2kx^2 - 3x + 4k = 0$ is twice the product of the roots.***

Solution: $2kx^2 - 3x + 4k = 0$

Here, $a = 2k, \quad b = -3, \; c = 4k$

Let $\propto, \beta$ be its roots, then:

$S = \propto + \beta = -\frac{b}{a} = -\frac{(-3)}{2k} = \frac{3}{2k}$

$P = \propto\beta = \frac{c}{a} = \frac{4k}{2k} = 2$

Now, applying the given condition

$S = 2P$ (putting values of S, P)

$\frac{3}{2k} = 2 \times 2 \Rightarrow \frac{3}{2 \times 2 \times 2} = k$

$$\frac{3}{8} = k$$

$$\therefore\ k = \frac{3}{8}$$

(ii) ***sum of the roots of the equation $x^2 + (3k - 7)x + 5k = 0$ is $\frac{3}{2}$ times the product of the roots.***

Solution: $x^2 + (3k - 7) + 5k = 0$

Here, $a = 1, b = 3k - 7, c = 5k$

Let α, β be its roots, then:

$$S = \alpha + \beta = -\frac{b}{a} = -\frac{(3k-7)}{1} = -3k + 7$$

$$P = \alpha\beta = \frac{c}{a} = \frac{5k}{1} = 5k$$

Now, applying the given condition

$$S = \frac{3}{2}P \qquad \text{(putting values of } S, P)$$

$$-3k + 7 = \frac{3}{2}(5k)$$

$$-3k + 7 = \frac{15k}{2}$$

$$-6k + 14 = 15k$$

$$-6k - 15k = -14$$

$$-21k = -14$$

$$k = \frac{-14}{-21} = \frac{2}{3}$$

3. ***Find k, if***

(i) ***sum of the squares of the roots of the equation $4kx^2 + 3kx - 8 = 0$ is 2.***

Solution: Let α, β be its roots of the equation

$$4kx^2 + 3kx - 8 = 0, \quad \text{then}$$

$$S = \alpha + \beta = -\frac{b}{a} = -\frac{3k}{4k}\ ,\quad P = \alpha\beta = \frac{c}{a} = \frac{-8}{4k}$$

$$= -\frac{3}{4}\ ,\qquad\qquad = \frac{-2}{k}$$

Now $\alpha^2+\beta^2 = 2$ (given condition)

$(\alpha + \beta)^2 - 2\alpha\beta = 2$

Putting values of $\alpha + \beta$, $\alpha\beta$

$$\left(-\frac{3}{4}\right)^2 - 2\left(\frac{-2}{k}\right) = 2$$

$$\frac{9}{16} + \frac{4}{k} = 2$$

$$\frac{4}{k} = 2 - \frac{9}{16}$$

$$\frac{4}{k} = \frac{32-9}{16}$$

$$\frac{4}{k} = \frac{23}{16}$$

$$k = \frac{4 \times 16}{23} = \frac{64}{23}$$

(ii) ***sum of the squares of the roots of the equation $x^2 - 2kx + (2k+1) = 0$ is 6.***

Solution: Let α, β be its roots of the equation

$x^2 - 2kx + (2k+1) = 0$, then

$$S = \alpha + \beta = -\frac{b}{a} = -\frac{(-2k)}{1}\ ,\quad P = \alpha\beta = \frac{c}{a} = \frac{2k+1}{1}$$

$$= 2k\ ,\qquad\qquad = 2k+1$$

Now, $\alpha^2+\beta^2 = 6$ (given condition)

$\Rightarrow$ $(\alpha + \beta)^2 - 2\alpha\beta = 6$

Putting values of $\alpha + \beta$, $\alpha\beta$

$$
\begin{aligned}
(2k)^2 - 2(2k+1) &= 6 \\
4k^2 - 4k - 2 &= 6 \\
4k^2 - 4k - 2 - 6 &= 0 \\
4k^2 - 4k - 8 &= 0 \qquad \text{(Dividing by 4.)} \\
k^2 - k - 2 &= 0 \\
k^2 + k - 2k - 2 &= 0 \\
k(k+1) - 2(k+1) &= 0 \\
(k+1)(k-2) &= 0 \\
k + 1 &= 0 \qquad \text{or} \qquad k - 2 = 0 \\
k &= -1 \qquad \text{or} \qquad k = 2
\end{aligned}
$$

4. Find p, if

(i) the roots of the equation $x^2 - x + p^2 = 0$ differ by unity.

Solution: Let $\propto$, and $\propto - 1$ be the roots of $x^2 - x + p^2 = 0$,

Then $\propto + \propto - 1 = -\dfrac{b}{a} = -\dfrac{(-1)}{1} = 1$

$2\propto - 1 = 1 \Rightarrow 2\propto = 1 + 1 \Rightarrow 2\propto = 2 \Rightarrow \propto = 1$ (i)

and $\propto(\propto - 1) = \dfrac{c}{a} = \dfrac{p^2}{1} = p^2$ or $\propto(\propto - 1) = p^2$ (ii)

Putting values of $\propto$ from equation (i) in equation (ii), we get

$$
\begin{aligned}
1(1-1) &= p^2 \\
1(0) &= p^2 \\
p^2 &= 0 \\
p &= 0
\end{aligned}
$$

(ii) the roots of the equation $x^2+3x+p-2 = 0$ differ by 2.

Solution: Let $\propto$, and $\propto - 2$ be the roots of $x^2 + 3x + p - 2 = 0$

Then $\propto + \propto - 2 = -\dfrac{b}{a} = -\dfrac{3}{1} = -3$

$2\propto - 2 = -3 \Rightarrow 2\propto = -3 + 2 \Rightarrow 2\propto = -1 \Rightarrow \propto = -\dfrac{1}{2}$ (i)

and $\propto(\propto-2)=\frac{c}{a}=\frac{p-2}{1}=p-2$ or $\propto(\propto-2)=p-2$(ii)

Putting values of $\propto$ from equation (i) in equation (ii), we get

$$-\frac{1}{2}\left(-\frac{1}{2}-2\right)=p-2$$
$$-\frac{1}{2}\left(\frac{-1-4}{2}\right)=p-2$$
$$-\frac{1}{2}\left(-\frac{5}{2}\right)=p-2$$
$$\frac{5}{4}=p-2$$
$$p=\frac{5}{4}+2$$
$$p=\frac{5+8}{4}$$
$$p=\frac{13}{4}$$

5. ***Find m, if***

(i) ***the roots of the equation $x^2-7x+3m-5=0$ satisfy the relation $3\alpha+2\beta=4$***

Solution: $x^2-7x+3m-5=0$

Let $\propto, \beta$ be its roots, then:

$S=\propto+\beta=-\frac{b}{a}=-\frac{(-7)}{1}=7$……….(i)

$P=\propto\beta=\frac{c}{a}=\frac{3m-5}{1}=3m-5$……(ii)

From (i) $\beta=7-\propto$

Now, $3\propto+2\beta=4$ (given)

Putting $\beta=7-\propto$ in it

$$3\propto + 2(7-\propto) = 4$$
$$3\propto + 14 - 2\propto = 4$$
$$3\propto - 2\propto = 4 - 14$$
$$\propto = -10$$

Now, $\propto + \beta = 7$ (from i)

Putting $\propto = -10$ in it

$$-10 + \beta = 7$$
$$\beta = 7 + 10 = 17$$

Now, $\propto\beta = 3m - 5$

Putting values of $\propto, \beta$ in it.

$$(-10)(17) = 3m - 5$$
$$-170 = 3m - 5$$
$$3m = -170 + 5$$
$$3m = -165$$
$$m = -\frac{165}{3} = -55$$

(ii) ***the roots of the equation $x^2 + 7x + 3m - 5 = 0$ satisfy the relation $3\alpha - 2\beta = 4$***

Solution: $x^2 + 7x + 3m - 5 = 0$

Let $\propto, \beta$ be the roots of this equation, then

$$S = \propto + \beta = -\frac{b}{a} = \frac{-7}{1} = -7 \qquad \text{(i)}$$

$$P = \propto\beta = \frac{c}{a} = \frac{3m-5}{1} = 3m - 5 \qquad \text{(ii)}$$

Now, $3\propto - 2\beta = 4$ (given)

$\beta = -7 - \propto$ from (i)

Putting $\beta = -7 - \propto$ in it

$$3\propto - 2(-7-\propto) = 4$$
$$3\propto + 14 + 2\propto = 4$$

$$5\alpha = -4 - 14$$
$$5\alpha = -10$$
$$\alpha = -2.$$

Putting $\alpha = -2$ in (i)

$$-2 + \beta = -7$$
$$\beta = -7 + 2 = -5$$

Putting values of α, β in it.

$$(-2)(-5) = 3m - 5$$
$$10 = 3m - 5$$
$$3m = 10 + 5$$
$$3m = 15$$
$$\therefore \quad m = 5$$

(iii) ***the roots of the equation $3x^2 - 2x + 7m + 2 = 0$ satisfy the relation $7\alpha - 3\beta = 18$***

Solution: $3x^2 - 2x + 7m + 2 = 0$

Let α, β be its roots, then

$$S \quad \alpha + \beta = \frac{b}{a} = -\frac{(-2)}{3} = \frac{2}{3} \qquad \text{(i)}$$

$$P \quad \alpha\beta = \frac{c}{a} = \frac{7m+2}{3} \qquad \text{(ii)}$$

$$7\alpha - 3\beta = 18 \qquad \text{(given relation) (iii)}$$

$$\beta = \frac{2}{3} - \alpha \quad \text{from (i)}$$

Putting $\beta = \frac{2}{3} - \alpha$ in (iii)

$$7\alpha - 3\left(\frac{2}{3} - \alpha\right) = 18$$
$$7\alpha - 2 + 3\alpha = 18$$
$$10\alpha = 18 + 2 = 20$$
$$\alpha = 2$$

Putting $\alpha = 2$ in (i)

$$2 + \beta = \frac{2}{3}$$

$$\beta = \frac{2}{3} - 2 = \frac{2-6}{3} = \frac{-4}{3}$$

Now, $\alpha\beta = \frac{7m+2}{3}$

Putting values of α, β in it.

$$2\left(-\frac{4}{3}\right) = \frac{7m+2}{3}$$

$$\frac{-8}{3} = \frac{7m+2}{3}$$

$$7m + 2 = -8$$

$$7m = -8 - 2$$

$$7m = -10$$

$$m = -\frac{10}{7}$$

6. ***Find m, if sum and product of the roots of the following equations is equal to a given number λ.***

(i) $(2m+3)x^2 + (7m-5)x + (3m-10) = 0$

Solution: $(2m+3)x^2 + (7m-5)x + (3m-10) = 0$

Let α, β be the roots of the equation, then

$$S = \alpha + \beta = -\frac{b}{a} = -\frac{(7m-5)}{(2m+3)} = \lambda \qquad \text{(i)}$$

and $$P = \alpha\beta = \frac{c}{a} = \frac{3m-10}{2m+3} = \lambda \qquad \text{(ii)}$$

$$-\frac{(7m-5)}{(2m+3)} = \frac{3m-10}{2m+3} \qquad (\text{each} = \lambda)$$

$$\Rightarrow \quad -(7m-5) = 3m - 10$$

$$-7m + 5 = 3m - 10$$

$$-7m - 3m = -10 - 5$$
$$-10m = -15$$
$$m = \frac{-15}{-10}$$

Thus, $m = \frac{3}{2}$

(ii) $\mathbf{4x^2 - (3 + 5m)x - (9m - 17) = 0}$

Solution: $4x^2 - (3 + 5m)x - (9m - 17) = 0$

Let α, β be the roots of the equation, then

$$S = \alpha + \beta = -\frac{b}{a} = -\frac{-(3+5m)}{4} = \lambda \quad \text{(given)}$$

and $$P = \alpha\beta = \frac{c}{a} = \frac{-(9m - 17)}{4} = \lambda \quad \text{(given)}$$

$$\frac{-(3 + 5m)}{4} = \frac{-(9m - 17)}{4} \quad \text{(each} = \lambda)$$

$$3 + 5m = -9m + 17$$
$$5m + 9m = 17 - 3$$
$$14m = 14$$
$$m = 1$$

EXERCISE 2.4

1. ***If α, β are the roots of the equation $x^2 + px + q = 0$, then evaluate***

(i) $\alpha^2 + \beta^2$ (ii) $\alpha^3\beta + \alpha\beta^3$ (iii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

Solution: $x^2 + px + q = 0$

Now, $S = \alpha + \beta = -\dfrac{b}{a} = \dfrac{-p}{1} = -p$

$P = \alpha\beta = \dfrac{c}{a} = \dfrac{q}{1} = q$

Part (i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

Putting values of $\alpha + \beta, \alpha\beta$, we get

$= (-p)^2 - 2q$

$= p^2 - 2q$

Part (ii) $\alpha^3\beta + \alpha\beta^3$

$= \alpha\beta(\alpha^2 + \beta^2)$

$= \alpha\beta[(\alpha + \beta)^2 - 2\alpha\beta]$

Putting values of $\alpha\beta, \alpha + \beta$, we get

$= q[(-p)^2 - 2q]$

$= q(p^2 - 2q)$

Part (iii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

$= \dfrac{\alpha^2 + \beta^2}{\alpha\beta}$

$= \dfrac{1}{\alpha\beta}[\alpha^2 + \beta^2]$

$= \dfrac{1}{\alpha\beta}[(\alpha + \beta)^2 - 2\alpha\beta]$

Putting values of $\alpha+\beta$, $\alpha\beta$, we get.

$$=\frac{1}{q}[(-p)^2-2q]$$

$$=\frac{1}{q}[p^2-2q]$$

2. ***If α, β are the roots of the equation $4x^2-5x+6=0$, then find the values of***

(i) $\frac{1}{\alpha}+\frac{1}{\beta}$ (ii) $\alpha^2\beta^2$

(iii) $\frac{1}{\alpha^2\beta}+\frac{1}{\alpha\beta^2}$ (iv) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

Solution: $4x^2-5x+6=0$

Thus, $\alpha+\beta=-\frac{b}{a}=-\frac{(-5)}{4}=\frac{5}{4}$

$$\alpha\beta=\frac{c}{a}=\frac{6}{4}=\frac{3}{2}$$

Part (i) $\frac{1}{\alpha}+\frac{1}{\beta}$

$$=\frac{\beta+\alpha}{\alpha\beta}$$

$$=\frac{\alpha+\beta}{\alpha\beta}$$

Putting values of $\alpha+\beta$, $\alpha\beta$, we get

$$=\frac{\frac{5}{4}}{\frac{3}{2}}=\frac{5}{4}\div\frac{3}{2}=\frac{5}{4_2}\times\frac{2^1}{3}$$

$$=\frac{5}{6}$$

Part (ii) $\alpha^2\beta^2$

$= (\alpha\beta)^2$

$= \left(\frac{3}{2}\right)^2$ Putting values of $\alpha\beta$, we get.

$= \frac{9}{4}$

Part (iii) $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$

$= \frac{\beta + \alpha}{\alpha^2\beta^2}$

$= \frac{1}{(\alpha\beta)^2}(\alpha + \beta)$

Putting values of $\alpha + \beta$, $\alpha\beta$, we get

$= \frac{1}{\left(\frac{3}{2}\right)^2} \left(\frac{5}{4}\right)$

$= \frac{1}{\frac{9}{4}} \left(\frac{5}{4}\right)$

$= \frac{1 \times 4}{9} \times \frac{5}{4} = \frac{5}{9}$

Part (iv) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

$= \frac{\alpha^3 + \beta^3}{\alpha\beta}$

$= \frac{(\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)}{\alpha\beta}$

$= \frac{(\alpha + \beta)[\alpha^2 + \beta^2 - \alpha\beta]}{\alpha\beta}$

$$= \frac{(\alpha + \beta)[(\alpha+\beta)^2 - 2\alpha\beta - \alpha\beta]}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)[(\alpha+\beta)^2 - 3\alpha\beta]}{\alpha\beta}$$

Putting values of $\alpha + \beta$, $\alpha\beta$, we get

$$= \frac{\frac{5}{4}\left[\left(\frac{5}{4}\right)^2 - 3\left(\frac{3}{2}\right)\right]}{\frac{3}{2}}$$

$$= \frac{5}{4}\left[\frac{25}{16} - \frac{9}{2}\right] \div \frac{3}{2}$$

$$= \frac{5}{4}\left[\frac{25 - 72}{16}\right] \times \frac{2}{3}$$

$$= \frac{5}{4} \times \frac{-47}{16_8} \times \frac{2^1}{3}$$

$$= -\frac{235}{96}$$

3. ***If α, β are the roots of the equation $lx^2 + mx + n = 0$ $(l \neq 0)$, then find the values of***

(i) $\alpha^3\beta^2 + \alpha^2\beta^3$ (ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

Solution: $lx^2 + mx + n = 0$

$$\alpha + \beta = -\frac{m}{l},\ \alpha\beta = \frac{n}{l}$$

Part (i) $\alpha^3\beta^2 + \alpha^2\beta^3$

$$= \alpha^2\beta^2(\alpha + \beta)$$

$$= (\alpha\beta)^2(\alpha + \beta)$$

Putting values of $\alpha + \beta, \alpha\beta$, we get

$$= \left(\frac{n}{l}\right)^2\left(\frac{-m}{l}\right)$$

$$= -\frac{mn^2}{l^3}$$

Part (ii) $$\frac{1}{\alpha^2}+\frac{1}{\beta^2}$$

$$= \frac{\beta^2+\alpha^2}{\alpha^2\beta^2}$$

$$= \frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$$

Putting values of $\alpha+\beta, \alpha\beta$, we get

$$= \frac{\left(\frac{m}{l}\right)^2-2\left(\frac{n}{l}\right)}{\left(\frac{n}{l}\right)^2}$$

$$= \left(\frac{m^2}{l^2}-\frac{2n}{l}\right)\div\left(\frac{n}{l}\right)^2$$

$$= \left(\frac{m^2}{l^2}-\frac{2n}{l}\right)\times\left(\frac{l}{n}\right)^2$$

$$= \left(\frac{m^2-2nl}{l^2}\right)\times\frac{l^2}{n^2}$$

$$= \frac{1}{n^2}(m^2-2nl)$$

Formation of a quadratic equation

If α and β are the roots of an equation, then equation is:

$x^2-(\alpha+\beta)x+\alpha\beta=0$

i.e. $x^2-Sx+P=0$

EXERCISE 2.5

1. Write the quadratic equations having following roots.

(a) $1, 5$

Solution:

S = Sum of the roots = $1 + 5 = 6$

P = Product of the roots = $(1)(5) = 5$

Equation is: $x^2 - Sx + P = 0$

Putting values of S and P, we get

$x^2 - 6x + 5 = 0$ (required eq.)

(b) $4, 9$

Solution:

S = Sum of the roots = $4 + 9 = 13$

P = Product of the roots = $4 \times 9 = 36$

Equation is: $x^2 - Sx + P = 0$

Putting values of S and P, we get

$x^2 - 13x + 36 = 0$ (reqd. eq.)

(c) $-2, 3$

Solution:

S = Sum of the roots = $-2 + 3 = 1$

P = Product of the roots = $(-2)(3) = -6$

Equation is: $x^2 - Sx + P = 0$

Putting values of S and P, we get

$x^2 - 1x + (-6) = 0$

$x^2 - x - 6 = 0$ (reqd. eq.)

(d) $0, -3$

Solution:

S = Sum of the roots = $0 + (-3) = 0 - 3 = -3$

P = Product of the roots = $(0)(-3) = 0$

Equation is: $x^2 - Sx + P = 0$

Putting values of S and P, we get

$$x^2 - (-3)x + 0 = 0$$
$$x^2 + 3x = 0 \qquad \text{(reqd. eq.)}$$

(e) $2, -6$

Solution:

$S =$ Sum of the roots $= 2 + (-6) = 2 - 6 = -4$

$P =$ Product of the roots $= (2)(-6) = -12$

Equation is: $x^2 - Sx + P = 0$

Putting values of S and P, we get

$$x^2 - (-4)x + (-12) = 0$$
$$x^2 + 4x - 12 = 0 \qquad \text{(reqd. eq.)}$$

(f) $-1, -7$

Solution:

$S =$ Sum of the roots $= (-1) + (-7) = -1 - 7 = -8$

$P =$ Product of the roots $= (-1)(-7) = 7$

Equation is: $x^2 - Sx + P = 0$

Putting values of S and P, we get

$$x^2 - (-8)x + 7 = 0$$
$$x^2 + 8x + 7 = 0 \qquad \text{(reqd. eq.)}$$

(g) $1 + i, 1 - i$

Solution:

$S =$ Sum of the roots $= 1 + i + 1 - i = 2$

$P =$ Product of the roots $= (1 + i)(1 - i)$

$= (1)^2 - (\sqrt{-1})^2$

$= 1 - (-1) = 1 + 1 = 2$

Equation is: $x^2 - Sx + P = 0$

Putting values of S and P, we get

$$x^2 - 2x + 2 = 0 \qquad \text{(reqd. eq.)}$$

(h) $3 + \sqrt{2}, 3 - \sqrt{2}$

Solution:

$S =$ Sum of the roots $= 3 + \sqrt{2} + 3 - \sqrt{2} = 6$

P Product of the roots $= (3+\sqrt{2})(3-\sqrt{2})$
$= (3)^2 - (\sqrt{2})^2$
$= 9 - 2 = 7$

Equation is: $x^2 - Sx + P = 0$

Putting values of S and P, we get

$x^2 - 6x + 7 = 0$ (reqd. eq.)

2. If α, β are the roots of the equation $x^2 - 3x + 6 = 0$. From equations whose roots are

(a) $2\alpha + 1, 2\beta + 1$ (b) α^2, β^2

(c) $\frac{1}{\alpha}, \frac{1}{\beta}$ (d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

(e) $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

Solution: Equation is: $x^2 - 3x + 6 = 0$

Thus, $\alpha + \beta = -\frac{(-3)}{1} = 3$, $\alpha\beta = \frac{6}{1} = 6$ (A)

(a) Roots of the required equation are: $2\alpha + 1, 2\beta + 1$

Thus,

S Sum of the roots $= 2\alpha + 1 + 2\beta + 1$
$= 2\alpha + 2\beta + 2$
$S = 2(\alpha + \beta) + 2$

Putting values of $\alpha + \beta$, we get

$= 2(3) + 2 = 6 + 2 = 8$ (i)

P = Product of the roots
$= (2\alpha + 1)(2\beta + 1)$
$= 4\alpha\beta + 2\alpha + 2\beta + 1$
$= 4\alpha\beta + 2(\alpha + \beta) + 1$

Putting values of $\alpha\beta$, $\alpha + \beta$ (from A)

$= 4(6) + 2(3) + 1$
$= 24 + 6 + 1 = 31$ (ii)

Equation is: $x^2 - Sx + P = 0$

(putting values of S and P from (i) and (ii))

$x^2 - 8x + 31 = 0$ is the reqd. eq

(b) ***Roots of the required equation are: α^2, β^2***

S = Sum of the roots $= \alpha^2 + \beta^2$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= (3)^2 - 2(6)$ (from A)

$= 9 - 12 = -3$ (i)

P = Product of the roots

$= \alpha^2\beta^2$

$= (\alpha\beta)^2$

$= (6)^2$ (from A)

$= 36$ (ii)

Required equation is:

$x^2 - Sx + P = 0$

Putting values of S and P from (i) and (ii)

$x^2 - (-3)x + 36 = 0$

$x^2 + 3x + 36 = 0$ is the reqd. eq.

(c) ***Roots of the required equation are: $\frac{1}{\alpha}, \frac{1}{\beta}$***

S = Sum of the roots $= \frac{1}{\alpha} + \frac{1}{\beta}$

$= \frac{\beta + \alpha}{\alpha\beta}$ or

$= \frac{\alpha + \beta}{\alpha\beta}$

Putting values of $\alpha\beta$, $\alpha + \beta$ from A

$S = \frac{3}{6} = \frac{1}{2}$ (i)

$P=$ Product of the roots $=\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)$

$$=\frac{1}{\alpha\beta}$$

$$=\frac{1}{6} \quad \text{(ii) (from } A\text{)}$$

Required equation is:

$$x^2-Sx+P=0$$

Putting values of S and P from (i) and (ii)

$$x^2-\frac{1}{2}x+\frac{1}{6}=0$$

multiplying by 6, both sides

$6x^2-3x+1=0$ is the reqd. eq.

(d) ***Roots of the required equation are: $\frac{\alpha}{\beta},\frac{\beta}{\alpha}$***

$S=$ Sum of the roots $=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

$$=\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

Putting values of $\alpha+\beta$, $\alpha\beta$, from A

$$=\frac{(3)^2-2(6)}{6}$$

$$=\frac{9-12}{6}$$

$$=-\frac{3}{6}=-\frac{1}{2} \quad \text{(i)}$$

P = Product of the roots $= \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$ (ii)

Required equation is:

$$x^2 - Sx + P = 0$$

Putting values of S and P from (i) and (ii)

$$x^2 - \left(-\frac{1}{2}\right)x + 1 = 0$$

$$x^2 + \frac{1}{2}x + 1 = 0 \quad \text{(multiplying by 2)}$$

$$2x^2 + x + 2 = 0 \quad \text{is the reqd. eq.}$$

(e) Roots of the required equation are: $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

S = Sum of the roots $= \alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta}$

$$= (\alpha + \beta) + \frac{\beta + \alpha}{\alpha\beta}$$

$$= (\alpha + \beta) + \frac{\alpha + \beta}{\alpha\beta}$$

Putting values of $\alpha + \beta$ and $\alpha\beta$ from (A)

$$S = 3 + \frac{3}{6}$$

$$= 3 + \frac{1}{2} = 3\frac{1}{2} = \frac{7}{2} \quad \text{(i)}$$

P = Product of the roots $= (\alpha + \beta)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$

$$= (\alpha + \beta)\left(\frac{\beta + \alpha}{\alpha\beta}\right) \quad \text{or}$$

$$(\alpha + \beta)\left(\frac{\alpha + \beta}{\alpha\beta}\right)$$

Putting values of $\alpha+\beta$ and $\alpha\beta$, from (A)

$$P = (3)\left(\frac{3}{6}\right) = \frac{3}{2} \qquad \text{(ii)}$$

Required equation is:

$$x^2 - Sx + P = 0$$

Putting values of S and P from (i) and (ii)

$$x^2 - \frac{7}{2}x + \frac{3}{2} = 0$$

(Multiplying by 2)

$$2x^2 - 7x + 3 = 0 \qquad \text{required eq.}$$

3. If α, β are the roots of the equation $x^2 + px + q = 0$. From equations whose roots are:

(a) α^2, β^2 **(b)** $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Solution:

Given equation is $x^2 + px + q = 0$ whose roots are α, β

$$\therefore \alpha+\beta = -\frac{p}{1} = -p, \qquad \alpha\beta = \frac{q}{1} = q \qquad (A)$$

(a) Roots of the required equation are: α^2, β^2

S = Sum of the roots $= \alpha^2 + \beta^2$

$$= (\alpha+\beta)^2 - 2\alpha\beta$$
$$= (-p)^2 - 2q \quad \text{(from } A\text{)}$$
$$= p^2 - 2q \ldots\ldots\ldots\ldots \text{(i)}$$

P = Product of the roots

$$= (\alpha^2)(\beta^2)$$
$$= (\alpha\beta)^2$$
$$= q^2 \ldots\ldots\ldots \text{(ii) (from } A\text{)}$$

Required equation is:

$$x^2 - Sx + P = 0$$

Putting the values of S and P from (i) and (ii) (reqd. eq.)

$$x^2 - (p^2 - 2q)x + q^2 = 0$$

(b) ***Roots of the required equation are:*** $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$

$$S = \text{Sum of the roots } = \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

$$= \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

Putting the values of $\alpha + \beta$ and $\alpha\beta$, from (A)

$$= \frac{(-p)^2 - 2q}{q}$$

$$= \frac{p^2 - 2q}{q} \quad \text{............(i)}$$

$$P = \text{Product of the roots} = \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$$

Required equation is:

$$x^2 - Sx + P = 0$$

Putting the values of S and P from (i) and (ii)

$$x^2 - \left(\frac{p^2 - 2q}{q}\right)x + 1 = 0$$

(multiplying by q)

$$qx^2 - (p^2 - 2q)x + q = 0 \qquad \text{(required eq.)}$$

EXERCISE 2.6

1. ***Use synthetic division to find the quotient and the remainder, when***

(i) $(x^2 + 7x - 1) \div (x + 1)$

Solution:

$$P(x) = x^2 + 7x - 1$$

Here, $x - a = x + 1$

$\therefore \quad a = -1$

Write co-efficients from dividend in a row.

$$\begin{array}{r|rrr} & 1 & 7 & -1 \\ -1 & \downarrow & -1 & -6 \\ \hline & 1 & 6 & -7 \end{array}$$

0

Depressed equation is $x + 6$ and remainder $= -7$

$$Q(x) = x + 6; \quad R = -7$$

(ii) $(4x^3 - 5x + 15) \div (x + 3)$

Solution:

$$P(x) = 4x^3 - 5x + 15$$
$$= 4x^3 + 0x^2 - 5x + 15$$

Here, $x - a = x + 3$

$a = -3$

Writing the co-efficients from $P(x)$ in a row.

$$\begin{array}{r|rrrr} & 4 & 0 & -5 & 15 \\ -3 & \downarrow & -12 & 36 & -93 \\ \hline & 4 & -12 & 31 & -78 \end{array}$$

$Q(x) = 4x^2 - 12x + 31$ and $R = -78$

(iii) $(x^3 + x^2 - 3x + 2) \div (x - 2)$

Solution:

$$P(x) = x^3 + x^2 - 3x + 2$$

Here, $x - a = x - 2$

$a = 2$

Writing the co-efficients from $P(x)$ in a row.

$$\begin{array}{r|rrr|r} & 1 & 1 & -3 & 2 \\ 2 & \downarrow & 2 & 6 & 6 \\ \hline & 1 & 3 & 3 & 8 \end{array}$$

$Q(x) = x^2 + 3x + 3$ and $R = 8$

2. ***Find the value of h using synthetic division, if***

(i) **3 is the zero of the polynomial $2x^3 - 3hx^2 + 9$**

Solution:

$$P(x) = 2x^3 - 3hx^2 + 9$$
$$= 2x^3 - 3hx^2 + 0x + 9$$

and $a = 3$

Thus.

$$\begin{array}{r|rrr|r} & 2 & -3h & 0 & 9 \\ 3 & \downarrow & 6 & 9h + 18 & 27h + 54 \\ \hline & 2 & -3h + 6 & -9h + 18 & -27h + 63 \end{array}$$

If 3 is zero of $P(x)$, then $R = 0$

Here. $R = -27h + 63 = 0$

$-27h + 63 = 0$

$27h = -63$

$h = \dfrac{-63}{-27} = \dfrac{7}{3}$

(ii) ***1 is the zero of the polynomial $x^3 - 2hx^2 + 11$***

Solution:

$$P(x) = x^3 - 2hx^2 + 11$$
$$= x^3 - 2hx^2 + 0x + 11$$

and $a = 1$

Thus,

$$\begin{array}{r|cccc}
 & 1 & -2h & 0 & 11 \\
1 & \downarrow & 1 & -2h+1 & -2h+1 \\
\hline
 & 1 & -2h+1 & -2h+1 & \multicolumn{1}{|c}{-2h+12}
\end{array}$$

If 1 is zero of the polynomial, then $R = 0$

Here, $R = -2h + 12 = 0$

$$-2h + 12 = 0$$
$$-2h = -12$$
$$h = \frac{-12}{-2} = 6$$

(iii) ***-1 is the zero of the polynomial $2x^3 + 5hx - 23$***

Solution:

$$P(x) = 2x^3 + 5hx - 23$$
$$= 2x^3 + 0x^2 + 5hx - 23$$

and $a = -1$

Thus,

$$\begin{array}{r|cccc}
 & 2 & 0 & 5h & -23 \\
-1 & \downarrow & -2 & 2 & -5h-2 \\
\hline
 & 2 & -2 & 5h+2 & \multicolumn{1}{|c}{-5h-25}
\end{array}$$

If -1 is zero of the polynomial, then $R = 0$

Here, $R = -5h - 25 = 0$

$$-5h - 25 = 0$$
$$-5h = 25$$
$$h = \frac{25}{-5} = -5$$

3. **Use synthetic division to find the values of l and m, if**

(i) **$(x + 3)$ and $(x - 2)$ are the factors of the polynomial $x^3 + 4x^2 + 2lx + m$**

Solution:

Here, $P(x) = x^3 + 4x^2 + 2lx + m$

and $x - a = x + 3$

Therefore, $a = -3$

By synthetic division

$$\begin{array}{r|cccc}
 & 1 & 4 & 2l & m \\
-3 & \downarrow & -3 & -3 & -6l+9 \\
\hline
 & 1 & 1 & 2l-3 & \multicolumn{1}{|c}{-6l+m+9}
\end{array}$$

Since -3 is zero of the polynomial, therefore, $R = 0$

$$R = -6l + m + 9 = 0$$

Thus, $-6l + m + 9 = 0$ ……………(i)

Again $x - a = x - 2$

$$a = 2$$

By synthetic division.

$$\begin{array}{r|cccc}
 & 1 & 4 & 2l & m \\
2 & \downarrow & 2 & 12 & 4l+24 \\
\hline
 & 1 & 6 & 2l+12 & \multicolumn{1}{|c}{4l+m+24}
\end{array}$$

Since 2 is zero of the polynomial, therefore, $R = 0$

$$R = 4l + m + 24 = 0$$

Thus, $4l + m + 24 = 0$ …………(ii)

Subtracting (ii) from (i)

$$-10l - 15 = 0$$

$$-10l = 15$$

$$l = \frac{15}{-10} = -\frac{3}{2}$$

Putting $l = -\frac{3}{2}$ in (i), we get

$$-6\left(-\frac{3}{2}\right) + m + 9 = 0$$

$$9 + m + 9 = 0$$

$$m = -9 - 9$$

$$m = -18$$

Thus, $\quad l = -\frac{3}{2}, \quad m = -18$

(ii) $(x-1)$ and $(x+1)$ are the factors of the polynomial $x^3 - 3lx^2 + 2mx + 6$

Solution:

Here, $\quad P(x) = x^3 - 3lx^2 + 2mx + 6$

and $\quad x - a = x - 1$

$\therefore \quad a = 1$

Using synthetic division

$$\begin{array}{c|cccc} & 1 & -3l & 2m & 6 \\ 1 & \downarrow & 1 & -3l+1 & 2m-3l+1 \\ \hline & 1 & -3l+1 & 2m-3l+1 & 2m-3l+7 \end{array}$$

Since 1 is zero of the polynomial, therefore, $R = 0$

$R = 2m - 3l + 7 = 0$

Thus $\quad 2m - 3l + 7 = 0$..................(i)

$P(x) = x^3 - 3lx^2 + 2mx + 6$

Again $\quad x - a = x + 1$

$\therefore \quad a = -1$

Using synthetic division.

$$\begin{array}{c|cccc} & 1 & -3l & 2m & 6 \\ -1 & \downarrow & -1 & 3l-1 & -2m-3l-1 \\ \hline & 1 & -3l-1 & 2m+3l+1 & -2m-3l+5 \end{array}$$

Since -1 is zero of the polynomial, therefore, $R = 0$

$R = -2m - 3l + 5 = 0$

Thus, $\quad -2m - 3l + 5 = 0$

Subtracting (ii) from (i), we get

$4m + 2 = 0$

$4m = -2$

$m = -\frac{2}{4} = -\frac{1}{2}$

Putting $m = -\frac{1}{2}$ in (i)

$$2\left(-\frac{1}{2}\right) - 3l + 7 = 0$$
$$-1 - 3l + 7 = 0$$
$$-3l = -7 + 1$$
$$-3l = -6$$
$$l = \frac{-6}{-3} = 2$$

Thus, $l = 2, \quad m = -\frac{1}{2}$

4. Solve by using synthetic division, if

(i) 2 is the root of the equation $x^3 - 28x + 48 = 0$

Solution:

Here, $P(x) = x^3 - 28x + 48$
$= x^3 + 0x^2 - 28x + 48$

and $a = 2$

Using synthetic division

$$\begin{array}{r|rrr|r} & 1 & 0 & -28 & 48 \\ 2 & \downarrow & 2 & 4 & -48 \\ \hline & 1 & 2 & -24 & 0 \end{array}$$

Depressed equation is:

$$x^2 + 2x - 24 = 0$$
$$x^2 + 6x - 4x - 24 = 0$$
$$x(x+6) - 4(x+6) = 0$$
$$(x+6)(x-4) = 0$$

$x + 6 = 0$ gives $x = -6$

or $x - 4 = 0$ gives $x = 4$

Roots are: 2, −6, 4

(ii) 3 is the root of the equation $2x^3 - 3x^2 - 11x + 6 = 0$

Solution:

Here, $P(x) = 2x^3 - 3x^2 - 11x + 6$

and $a = 3$

By synthetic division

$$\begin{array}{c|cccc} & 2 & -3 & -11 & 6 \\ 3 & \downarrow & 6 & 9 & -6 \\ \hline & 2 & 3 & -2 & \multicolumn{1}{|c}{0} \end{array}$$

Depressed equation is:

$$\begin{aligned} 2x^2 + 3x - 2 &= 0 \\ 2x^2 + 4x - x - 2 &= 0 \\ 2x(x+2) - 1(x+2) &= 0 \\ (x+2)(2x-1) &= 0 \\ x + 2 &= 0 \quad \text{gives} \quad x = -2 \\ \text{or} \quad 2x - 1 &= 0 \quad \text{gives} \quad x = \frac{1}{2} \end{aligned}$$

Roots are: $3, -2, \frac{1}{2}$

(iii) -1 is the root of the equation $4x^3 - x^2 - 11x - 6 = 0$

Solution:

Here, $P(x) = 4x^3 - x^2 - 11x - 6$

and $a = -1$

By synthetic division

$$\begin{array}{c|cccc} & 4 & -1 & -11 & -6 \\ -1 & \downarrow & -4 & 5 & 6 \\ \hline & 4 & -5 & -6 & \multicolumn{1}{|c}{0} \end{array}$$

Depressed equation is:

$$\begin{aligned} 4x^2 - 5x - 6 &= 0 \\ 4x^2 - 8x + 3x - 6 &= 0 \\ 4x(x-2) + 3(x-2) &= 0 \\ (x-2)(4x+3) &= 0 \end{aligned}$$

$$x-2 = 0 \quad \text{gives} \quad x = 2$$

or $$4x+3 = 0 \quad \text{gives} \quad x = -\frac{3}{4}$$

Roots are: $-1, 2, -\frac{3}{4}$

5. Solve by using synthetic division, if

(i) 1 and 3 are the roots of the equation $x^4 - 10x^2 + 9 = 0$

Solution:

Here, $P(x) = x^4 - 10x^2 + 9$
$= x^4 + 0x^3 - 10x^2 + 0x + 9$

and $a = 1$

By synthetic division

$$\begin{array}{c|ccccc} & 1 & 0 & -10 & 0 & 9 \\ 1 & \downarrow & 1 & 1 & -9 & -9 \\ \hline & 1 & 1 & -9 & -9 & \boxed{0} \end{array}$$

Depressed equation is:

$$x^3 + x^2 - 9x - 9 = 0$$

3 is again a root of the equation.

Here, $a = 3$

$$\begin{array}{c|cccc} & 1 & 1 & -9 & -9 \\ 3 & \downarrow & 3 & 12 & 9 \\ \hline & 1 & 4 & 3 & \boxed{0} \end{array}$$

Depressed equation is

$$x^2 + 4x + 3 = 0$$
$$x^2 + x + 3x + 3 = 0$$
$$x(x+1) + 3(x+1) = 0$$
$$(x+1)(x+3) = 0$$
$$x+1 = 0 \quad \text{gives} \quad x = -1$$

or $$x+3 = 0 \quad \text{gives} \quad x = -3$$

Roots are: $1, 3, -1, -3$

(ii) 3 and – 4 are the roots of the equation
$$x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$$

Solution:

Here, $P(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$

and $a = 3$

By synthetic division

$$\begin{array}{r|rrrr|r} & 1 & 2 & -13 & -14 & 24 \\ 3 & \downarrow & 3 & 15 & 6 & -24 \\ \hline & 1 & 5 & 2 & -8 & 0 \end{array}$$

Depressed equation is:
$$x^3 + 5x^2 + 2x - 8 = 0$$

Again $a = -4$

Therefore,
$$\begin{array}{r|rrr|r} & 1 & 5 & 2 & -8 \\ -4 & \downarrow & -4 & -4 & 8 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

Depressed equation is:
$$\begin{aligned} x^2 + x - 2 &= 0 \\ x^2 + 2x - x - 2 &= 0 \\ x(x+2) - 1(x+2) &= 0 \\ (x+2)(x-1) &= 0 \\ x + 2 &= 0 \quad \text{gives} \quad x = -2 \\ \text{or} \quad x - 1 &= 0 \quad \text{gives} \quad x = 1 \end{aligned}$$

Roots are: 3, – 4, – 2, 1

Simultaneous Equations

A system of equations having a common solution is called a system of simultaneous equations.

EXERCISE 2.7

Solve the following simultaneous equations.

1. $x + y = 5$; $x^2 - 2y - 4 = 0$

Solution:

$$x + y = 5 \quad \text{.....(i)}$$
$$x^2 - 2y - 14 = 0 \quad \text{.....(ii)}$$

From (i) $x = 5 - y$

Putting $x = 5 - y$ in (ii), we get

$$(5 - y)^2 - 2y - 14 = 0$$
$$25 + y^2 - 10y - 2y - 14 = 0$$
$$y^2 - 12y + 25 - 14 = 0$$
$$y^2 - 12y + 11 = 0$$
$$y^2 - y - 11y + 11 = 0$$
$$y(y - 1) - 11(y - 1) = 0$$
$$(y - 1)(y - 11) = 0$$
$$y - 1 = 0 \quad \text{gives} \quad y = 1$$
or $$y - 11 = 0 \quad \text{gives} \quad y = 11$$

Put $y = 1$ in (i), we get

$$x + 1 = 5$$
$$x = 5 - 1 = 4$$

One pair is (4, 1)

When $y = 11$

Put $y = 11$ in (i), we get

$$x + 11 = 5$$
$$x = 5 - 11 = -6$$

Second pair is $(-6, 11)$

$$\text{Sol. Set} = \{(4, 1), (-6, 11)\}$$

2. $3x - 2y = 1$; $x^2 + xy - y^2 = 1$

Solution:

$$3x - 2y = 1 \quad \text{(i)}$$
$$x^2 + xy - y^2 = 1 \quad \text{(ii)}$$

From (i) $3x = 1 + 2y$

$$x = \frac{1+2y}{3}$$

Putting this value of x in eq. (ii)

$$\left(\frac{1+2y}{3}\right)^2 + \left(\frac{1+2y}{3}\right)(y) - y^2 = 1$$

$$\frac{1+4y^2+4y}{9} + \frac{y+2y^2}{3} - y^2 = 1$$

Multiplying by 9, both sides

$$1 + 4y^2 + 4y + 3(y+2y^2) - 9y^2 = 9$$

$$1 + 4y^2 + 4y + 3y + 6y^2 - 9y^2 - 9 = 0$$

$$y^2 + 7y - 8 = 0$$

$$y^2 - y + 8y - 8 = 0$$

$$y(y-1) + 8(y-1) = 0$$

$$(y-1)(y+8) = 0$$

$y - 1 = 0$ gives $y = 1$

$y + 8 = 0$ gives $y = -8$

When $y = 1$, put $y = 1$ in equation (i)

$$3x - 2(1) = 1$$

$$3x - 2 = 1$$

$$3x = 1 + 2$$

$$3x = 3$$

$$x = 1$$

One pair is (1, 1)

When $y = -8$, put $y = -8$ in equation (i)

$$3x - 2(-8) = 1$$

$$3x + 16 = 1$$

$$3x = 1 - 16$$

$$3x = -15$$

$$x = \frac{-15}{3} = -5$$

Second pair is $(-5, -8)$

$$\text{Sol. set} = \{(1, 1), (-5, -8)\}$$

3. $x - y = 7;\quad \frac{2}{x} - \frac{5}{y} = 2$

Solution:

$$x - y = 7 \ldots\ldots\ldots(i)$$

$$\frac{2}{x} - \frac{5}{y} = 2 \ldots\ldots\ldots(ii)$$

Multiplying by xy

$$2y - 5x = 2xy \ldots\ldots(iii)$$

$$x - y = 7 \quad \text{(from i)}$$

$$x = 7 + y$$

Putting $x = 7 + y$ in (iii), we get

$$2y - 5(7 + y) = 2(7 + y)(y)$$

$$2y - 35 - 5y = 2(7y + y^2)$$

$$-3y - 35 = 14y + 2y^2$$

$$2y^2 + 14y + 3y + 35 = 0$$

$$2y^2 + 17y + 35 = 0$$

$$2y^2 + 7y + 10y + 35 = 0$$

$$y(2y + 7) + 5(2y + 7) = 0$$

$$(2y + 7)(y + 5) = 0$$

$$2y + 7 = 0 \quad \text{gives} \quad y = -\frac{7}{2}$$

$$y + 5 = 0 \quad \text{gives} \quad y = -5$$

Putting $y = -\frac{7}{2}$ in (i), we get

$$x - \left(-\frac{7}{2}\right) = 7$$

$$x + \frac{7}{2} = 7$$

$$x = 7 - \frac{7}{2}$$

$$x = \frac{14-7}{2}$$

$$x = \frac{7}{2}$$

One pair is $\left(\frac{7}{2}, -\frac{7}{2}\right)$

Putting $y = -5$ in (i), we get

$$x - (-5) = 7$$
$$x + 5 = 7$$
$$x = 7 - 5$$
$$x = 2$$

Second pair is $(2, -5)$

$$\textbf{Sol. set} = \left\{(2,-5)\left(\frac{7}{2}, -\frac{7}{2}\right)\right\}$$

4. $x + y = a - b$; $\frac{a}{x} - \frac{b}{y} = 2$

Solution:

$$x + y = a - b \ldots\ldots(i)$$

$$\frac{a}{x} - \frac{b}{y} = 2 \ldots\ldots\ldots(ii)$$

(Multiplying by xy, both sides)

$$ay - bx = 2xy$$

from (i) $x = a - b - y$ ....(iv)

Putting this value of x in (iii), we get.

$$ay - b(a - b - y) = 2(a - b - y)(y)$$
$$ay - ab + b^2 + by = 2ay - 2by - 2y^2$$
$$2y^2 + ay + by - 2ay + 2by - ab + b^2 = 0$$
$$2y^2 - ay + 3by - ab + b^2 = 0$$
$$2y^2 - (a - 3b)y - (ab - b^2) = 0$$
$$y = \frac{(a-3b) \pm \sqrt{[-(a-3b)^2] + 4(2)(ab - b^2)}}{2(2)}$$

$$y = \frac{(a-3b) \pm \sqrt{a^2+9b^2-6ab+8ab-8b^2}}{4}$$

$$y = \frac{(a-3b) \pm \sqrt{a^2+2ab+b^2}}{4}$$

$$y = \frac{(a-3b) \pm \sqrt{(a+b)^2}}{4}$$

$$y = \frac{(a-3b) \pm (a+b)}{4}$$

$$y = \frac{(a-3b) + (a+b)}{4}$$

or $$y = \frac{(a-3b) - (a+b)}{4}$$

$$y = \frac{a-3b+a+b}{4} \quad , \quad y = \frac{a-3b-a-b}{4}$$

$$y = \frac{2a-2b}{4} \quad , \quad y = \frac{-4b}{4}$$

$$y = \frac{2(a-b)}{4} \quad , \quad y = -b$$

$$y = \frac{a-b}{2}$$

When $y = \dfrac{a-b}{2}$

Put it in (iv)

$$x = a-b-\left(\frac{a-b}{2}\right)$$

$$x = \frac{2a-2b-a+b}{2}$$

$$x = \frac{a-b}{2}$$

One pair is $\left(\dfrac{a-b}{2}, \dfrac{a-b}{2}\right)$

When $y = -b$

Put it in (iv)

$$x = a - b - (-b)$$
$$x = a - b + b = a$$

Second pair is $(a, -b)$

Sol. set $= \left\{(a, -b), \left(\dfrac{a-b}{2}, \dfrac{a-b}{2}\right)\right\}$

5. $x^2 + (y-1)^2 = 10$; $x^2 + y^2 + 4x = 1$

Solution:

$$x^2 + (y-1)^2 = 10 \quad \text{.......... (i)}$$
$$x^2 + y^2 + 4x = 1 \quad \text{.......... (ii)}$$

from (i) $\quad x^2 + y^2 - 2y + 1 = 10$

$$x^2 + y^2 - 2y = 10 - 1$$
$$x^2 + y^2 - 2y = 9 \quad \text{......... (iii)}$$

from (ii) $\quad x^2 + y^2 + 4x = 1$

Subtracting (ii) from (iii)

$$-2y - 4x = 8$$
$$2y + 4x = -8$$
$$y + 2x = -4 \quad \text{(Dividing by 2)}$$
$$y = -4 - 2x \quad \text{........ (A)}$$

Putting $y = -4 - 2x$ in (ii)

$$x^2 + (-4 - 2x)^2 + 4x = 1$$
$$x^2 + 16 + 4x^2 + 16x + 4x = 1$$
$$5x^2 + 20x + 15 = 0$$
$$x^2 + 4x + 3 = 0 \quad \text{(Dividing by 5)}$$
$$x^2 + x + 3x + 3 = 0$$
$$x(x+1) + 3(x+1) = 0$$
$$(x+1)(x+3) = 0$$

$$
\begin{aligned}
& x+1 = 0 \quad \text{gives} \quad x=-1\\
\text{or}\quad & x+3 = 0 \quad \text{gives} \quad x=-3\\
\text{When}\quad & x = -1 \text{ then from (A)}\\
& y = -4-2(-1)\\
& y = -4+2=-2
\end{aligned}
$$

One pair is $(-1,-2)$

$$
\begin{aligned}
\text{When}\quad & x = -3, \text{ then from (A)}\\
& y = -4-2(-3)\\
& y = -4+6=2
\end{aligned}
$$

Second pair is $(-3,2)$

$$\textbf{Sol. set} = \{(-1,-2),(-3,2)\}$$

6. $(x+1)^2+(y+1)^2=5\ ;\ (x+2)^2+y^2=5$

Solution:

$$
\begin{aligned}
(x+1)^2+(y+1)^2 &= 5 \quad \ldots\ldots\ldots\text{(i)}\\
(x+2)^2+y^2 &= 5 \quad \ldots\ldots\ldots\text{(ii)}\\
x^2+2x+1+y^2+2y+1 &= 5 \quad \text{from (i)}\\
x^2+y^2+2x+2y+2 &= 5\\
x^2+y^2+2x+2y &= 3 \quad \ldots\ldots\ldots\text{(A)}\\
x^2+4x+4+y^2 &= 5 \quad \text{from (ii)}\\
x^2+y^2+4x &= 1 \quad \ldots\ldots\ldots\text{(B)}
\end{aligned}
$$

$(A - B)$ gives

$$
\begin{aligned}
-2x-2y &= 2\\
x-y &= -1 \quad \text{(Dividing by } -2)\\
x &= y-1 \quad \ldots\ldots\ldots\ldots\text{(iii)}
\end{aligned}
$$

Putting this value of x in (A)

$$
\begin{aligned}
(y-1)^2+y^2+2(y-1)+2y &= 3\\
y^2-2y+1+y^2+2y-2+2y &= 3\\
2y^2+2y+1-2-3 &= 0\\
2y^2+2y-4 &= 0\\
y^2+y-2 &= 0 \quad \text{(Dividing by 2)}
\end{aligned}
$$

$$y^2 + 2y - y - 2 = 0$$
$$y(y+2) - 1(y+2) = 0$$
$$(y+2)(y-1) = 0$$
$$y+2 = 0 \quad \text{gives} \quad y = -2$$
or $$y - 1 = 0 \quad \text{gives} \quad y = 1$$
When $y = -2$, then from (iii)
$$x = -2 - 1 = -3$$
One pair is $(-3, -2)$

When $y = 1$, then from (iii)
$$x = 1 - 1 = 0$$
Second pair is $(0,1)$

Sol. set $= \{(0,1), (-3, -2)\}$

7. $x^2 + 2y^2 = 22$, $5x^2 + y^2 = 29$

Solution:

$$x^2 + 2y^2 = 22 \quad \ldots\ldots\text{(i)}$$
$$5x^2 + y^2 = 29 \quad \ldots\ldots\text{(ii)}$$
multiplying (i) by 5
$$5x^2 + 10y^2 = 110 \quad \ldots\ldots\text{(iii)}$$
Subtracting (ii) from (iii)
$$-9y^2 = -81$$
$$y^2 = 9 \quad \text{(Dividing by } -9\text{)}$$
Thus, $y = \pm 3$

When $y = 3$, put it in (i)
$$x^2 + 2(3)^2 = 22$$
$$x^2 + 18 = 22$$
$$x^2 = 22 - 18$$
$$x^2 = 4$$
Thus, $x = \pm 2$

One pair is $(\pm 2, 3)$

When $y = -3$, then from (i)

$$x^2 + 2(-3)^2 = 22$$
$$x^2 + 18 = 22$$
$$x^2 = 22 - 18$$
$$x^2 = 4$$

Thus, $x = \pm 2$

Second pair is $(\pm 2, -3)$

Sol. set $= \{(\pm 2, \pm 3)\}$

8. $4x^2 - 5y^2 = 6$, $3x^2 + y^2 = 14$

Solution:

$$4x^2 - 5y^2 = 6 \quad \ldots\ldots\ldots (i)$$
$$3x^2 + y^2 = 14 \quad \ldots\ldots\ldots (ii)$$

multiplying (ii) by 5,

$$15x^2 + 5y^2 = 70 \quad \ldots\ldots\ldots (iii)$$

Adding (i) and (iii)

$$19x^2 = 76$$
$$x^2 = 4$$
$$x = \pm 2$$

When $x = 2$, (ii) gives

$$3(2)^2 + y^2 = 14$$
$$12 + y^2 = 14$$
$$y^2 = 14 - 12$$
$$y^2 = 2$$
$$y = \pm\sqrt{2}$$

One pair is $(2, \pm\sqrt{2})$

When $x = -2$, (ii) gives

$$3(-2)^2 + y^2 = 14$$
$$12 + y^2 = 14$$
$$y^2 = 14 - 12$$
$$y^2 = 2$$
$$y = \pm\sqrt{2}$$

Second pair is $\left(-2, \pm\sqrt{2}\right)$

Sol. set $= \left\{\left(\pm 2, \pm\sqrt{2}\right)\right\}$

9. $7x^2 - 3y^2 = 4$, $2x^2 + 5y^2 = 7$

Solution:

$$7x^2 - 3y^2 = 4 \quad \text{......... (i)}$$
$$2x^2 + 5y^2 = 7 \quad \text{.........(ii)}$$

multiplying (i) by 5 and (ii) by 3.

$$35x^2 - 15y^2 = 20 \quad \text{.........(A)}$$
$$6x^2 + 15y^2 = 21 \quad \text{......... (B)}$$

(A + B) gives

$$41x^2 = 41$$
$$x^2 = 1$$

Thus, $x = \pm 1$

When $x = 1$, (ii) gives

$$2(1)^2 + 5y^2 = 7$$
$$2 + 5y^2 = 7$$
$$5y^2 = 7 - 2$$
$$5y^2 = 5$$
$$y^2 = 1$$
$$y = \pm 1$$

One pair is $(1, \pm 1)$

When $x = -1$, (ii) gives

$$2(-1)^2 + 5y^2 = 7$$
$$2 + 5y^2 = 7$$
$$5y^2 = 7 - 2$$
$$5y^2 = 5$$
$$y^2 = 1$$
$$y = \pm 1$$

Second pair is $(-1, \pm 1)$

Sol. set $= \{(\pm 1, \pm 1)\}$

10. $x^2 + 2y^2 = 3$, $x^2 + 4xy - 5y^2 = 0$

Solution:

$$x^2 + 2y^2 = 3 \quad \text{.......... (i)}$$

$$x^2 + 4xy - 5y^2 = 0 \quad \text{..........(ii)}$$
$$x^2 + 5xy - xy - 5y^2 = 0 \quad \text{from (ii)}$$
$$x(x + 5y) - y(x + 5y) = 0$$
$$(x + 5y)(x - y) = 0$$
$$x - y = 0 \text{ gives } x = y \text{........ (A)}$$
$$x + 5y = 0 \text{ gives } x = -5y \text{.... (B)}$$

When $x = y$, (i) gives

$$(y)^2 + 2y^2 = 3$$
$$y^2 + 2y^2 = 3$$
$$3y^2 = 3$$
$$y^2 = 1$$
$$y = \pm 1$$

then from (A), put $y = \pm 1$ in it.

$$x = \pm 1$$

One pair is $(\pm 1, \pm 1)$

When $x = -5y$ (i), gives

$$(-5y)^2 + 2y^2 = 3$$
$$25y^2 + 2y^2 = 3$$
$$27y^2 = 3$$
$$y^2 = \frac{3}{27}$$
$$y^2 = \frac{1}{9} \quad \text{(taking sq. root)}$$
$$y = \pm\frac{1}{3}$$

Put $y = \pm\frac{1}{3}$ in (B)

$$x = -5\left(\pm\frac{1}{3}\right)$$
$$= \left(\mp\frac{5}{3}\right)$$

Second pair is $\left(\mp\frac{5}{3},\pm\frac{1}{3}\right)$

$$\text{Sol. set} = \left\{(\pm1,\pm1),\left(\mp\frac{5}{3},\pm\frac{1}{3}\right)\right\}$$

11. $3x^2 - y^2 = 26$, $3x^2 - 5xy - 12y^2 = 0$

Solution:

$$3x^2 - y^2 = 26 \quad \ldots\ldots(i)$$
$$3x^2 - 5xy - 12y^2 = 0 \quad \ldots\ldots(ii)$$
$$3x^2 - 9xy + 4xy - 12y^2 = 0 \quad \text{from (ii)}$$
$$3x(x - 3y) + 4y(x - 3y) = 0$$
$$(x - 3y)(3x + 4y) = 0$$
$$x - 3y = 0 \text{ gives } x = 3y \ldots\ldots(A)$$
$$3x + 4y = 0 \text{ gives } x = -\frac{4y}{3} \ldots(B)$$

When $x = 3y$, (i) becomes

$$3(3y)^2 - y^2 = 26$$
$$3(9y^2) - y^2 = 26$$
$$27y^2 - y^2 = 26$$
$$26y^2 = 26$$
$$y^2 = 1$$
$$y = \pm 1$$

Put $y = \pm 1$ in (A)

$$x = 3(\pm1) = \pm 3$$

This gives ordered pair as $(\pm 3, \pm 1)$

When $x = \frac{-4y}{3}$ (i) becomes

$$3\left(-\frac{4y}{3}\right)^2 - y^2 = 26$$
$$3\left(\frac{16y^2}{9}\right) - y^2 = 26$$

$$\frac{16y^2}{3} - y^2 = 26$$

$$16y^2 - 3y^2 = 78 \quad \text{(multiplying by 3)}$$

$$13y^2 = 78$$

$$y^2 = \frac{78}{13}$$

$$y^2 = 6$$

$$y = \pm\sqrt{6}$$

Put $y = \pm\sqrt{6}$ in (B)

$$x = -\frac{4(\pm\sqrt{6})}{3}$$

$$x = \mp\frac{4\sqrt{6}}{3}$$

This gives ordered pair as $\left(\mp\frac{4\sqrt{6}}{3}, \pm\sqrt{6}\right)$

$$\textbf{Sol. set} = \left\{(\pm 3, \pm 1), \left(\mp\frac{4\sqrt{6}}{3}, \pm\sqrt{6}\right)\right\}$$

12. $x^2 + xy = 5$, $y^2 + xy = 3$

Solution:

$$x^2 + xy = 5 \quad \text{...... (i)}$$

$$y^2 + xy = 3 \quad \text{...... (ii)}$$

$$x(x + y) = 5 \quad \text{from (i) ...... (A)}$$

$$y(y + x) = 3 \quad \text{from (ii)}$$

or

$$y(x + y) = 3 \quad \text{.................. (B)}$$

$(A \div B)$ gives

$$\frac{x}{y} = \frac{5}{3}$$

$$x = \frac{5}{3}y \quad \text{............(C)}$$

Putting $x = \frac{5}{3}y,$ in (i)

$$\left(\frac{5}{3}y\right)^2 + \left(\frac{5}{3}y\right)y = 5$$

$$\frac{25}{9}y^2 + \frac{5}{3}y^2 = 5$$

$$25y^2 + 15y^2 = 45 \quad \text{(multiplying by 9)}$$

$$5y^2 + 3y^2 = 9 \quad \text{(Dividing by 5)}$$

$$8y^2 = 9$$

$$y^2 = \frac{9}{8}$$

$$y = \pm\sqrt{\frac{9}{8}} = \pm\frac{3}{2\sqrt{2}}$$

When $y = +\frac{3}{2\sqrt{2}}$ (C) gives

$$x = \left(\frac{5}{3}\right)\left(\frac{3}{2\sqrt{2}}\right) = \frac{5}{2\sqrt{2}}$$

One pair is $\left(\frac{5}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}\right)$

When $x = \frac{-3}{2\sqrt{2}}$ C gives

$$x = \frac{5}{3}\left(-\frac{3}{2\sqrt{2}}\right)$$

$$= -\frac{5}{2\sqrt{2}}$$

Second pair is $\left(-\frac{5}{2\sqrt{2}}, -\frac{3}{2\sqrt{2}}\right)$

Sol. Set $= \left(-\frac{5}{2\sqrt{2}}, -\frac{3}{2\sqrt{2}}\right)$

13. $x^2 - 2xy = 7$, $xy + 3y^2 = 2$

Solution:

$$x^2 - 2xy = 7 \quad \ldots\ldots\ldots (i)$$
$$xy + 3y^2 = 2 \quad \ldots\ldots\ldots (ii)$$

multiplying (i) by 2 and (ii) by 7

$$2x^2 - 4xy = 14 \quad \ldots\ldots\ldots (A)$$
$$7xy + 21y^2 = 14 \quad \ldots\ldots\ldots (B)$$

$(A \quad B)$ gives

$$2x^2 - 11xy - 21y^2 = 0$$
$$2x^2 - 14xy + 3xy - 21y^2 = 0$$
$$2x(x - 7y) + 3y(x - 7y) = 0$$
$$(x - 7y)(2x + 3y) = 0$$
$$x - 7y = 0 \text{ gives } x = 7y \ldots\ldots (C)$$
$$2x + 3y = 0 \text{ gives } x = \frac{-3y}{2} \ldots (D)$$

Put $x = 7y$ in (i), then we get

$$(7y)^2 - 2(7y)y = 7$$
$$49y^2 - 14y^2 = 7$$
$$35y^2 = 7$$
$$y^2 = \frac{7}{35}$$
$$y^2 = \frac{1}{5}$$
$$y = \pm\frac{1}{\sqrt{5}}$$

Putting $y = \pm\frac{1}{\sqrt{5}}$ in (C)

$$x = 7\left(\pm\frac{1}{\sqrt{5}}\right)$$
$$= \pm\frac{7}{\sqrt{5}}$$

One pair is $\left(\pm\frac{7}{\sqrt{5}}, \pm\frac{1}{\sqrt{5}}\right)$

From D, $x = -\frac{3y}{2}$

Putting this value in (i)

$$\left(-\frac{3y}{2}\right)^2 - 2\left(\frac{-3y}{2}\right)(y) = 7$$

$$\frac{9y^2}{4} + 3y^2 = 7$$

$$9y^2 + 12y^2 = 28 \quad \text{(multiplying by 4)}$$

$$21y^2 = 28$$

$$y^2 = \frac{28}{21}$$

$$y^2 = \frac{4}{3}$$

$$y = \pm\frac{2}{\sqrt{3}}$$

Putting $y = \pm\frac{2}{\sqrt{3}}$ in (D)

$$x = \left(-\frac{3}{2}\right)\left(\pm\frac{2}{\sqrt{3}}\right)$$

$$x = \mp\sqrt{3}$$

Second pair is $\left(\mp\sqrt{3}, \pm\frac{2}{\sqrt{3}}\right)$

$$\text{Sol. Set} = \left\{\left(\pm\frac{7}{\sqrt{5}}, \pm\frac{1}{\sqrt{5}}\right), \left(\mp\sqrt{3}, \pm\frac{2}{\sqrt{3}}\right)\right\}$$

EXERCISE 2.8

1. ***The product of two positive consecutive numbers is 182. Find the numbers.***

Solution: Let the numbers be x, $x + 1$

Then,
$$
\begin{aligned}
(x)(x+1) &= 182\\
x^2 + x - 182 &= 0\\
x^2 + 14x - 13x - 182 &= 0\\
x(x+14) - 13(x+14) &= 0\\
(x-13)(x+14) &= 0\\
x - 13 &= 0 \quad \text{gives} \quad x = 13
\end{aligned}
$$

Numbers are:
$$
\begin{aligned}
x &= 13\\
x + 1 &= 13 + 1 = 14
\end{aligned}
$$

Now, $x + 14 = 0$ gives $x = -14$

We ignore this value.

2. **The sum of the squares of three positive consecutive numbers is 77. Find them.**

Solution: Let the numbers be x, $x + 1$, $x + 2$

Applying the given condition.
$$
\begin{aligned}
(x)^2 + (x+1)^2 + (x+2)^2 &= 77\\
x^2 + x^2 + 2x + 1 + x^2 + 4x + 4 &= 77\\
3x^2 + 6x + 5 &= 77\\
3x^2 + 6x + 5 - 77 &= 0\\
3x^2 + 6x - 72 &= 0\\
x^2 + 2x - 24 &= 0 \quad \text{(Dividing by 3)}\\
x^2 + 6x - 4x - 24 &= 0\\
x(x+6) - 4(x+6) &= 0
\end{aligned}
$$

$$(x-4)(x+6) = 0$$

$$x-4 = 0 \quad \text{gives} \quad x = 4$$

Numbers are: $x = 4$

$$x+1 = 4+1 = 5$$

$$x+2 = 4+2 = 6$$

4, 5, 6

$$x+6 = 0$$

$x = -6$, we ignore it.

3. ***The sum of five times a number and the square of the number is 204. Find the number.***

Solution: Let the number be x

According to the given condition.

$$x^2+5x = 204$$

$$x^2+5x-204 = 0$$

$$x^2+17x-12x-204 = 0$$

$$x(x+17)-12(x+17) = 0$$

$$(x+17)(x-12) = 0$$

$$x-12 = 0 \quad \text{gives} \quad x = 12$$

$$\text{or} \quad x+17 = 0 \quad \text{gives} \quad x = -17$$

Number is 12 or -17.

4. ***The product of five less than three times a certain number and one less than four times the number is 7. Find the number.***

Solution: Let the number be x.

According the given condition.

$$(3x-5)(4x-1) = 7$$

$$12x^2-23x+5 = 7$$

$$12x^2-23x+5-7 = 0$$

$$12x^2 - 23x - 2 = 0$$
$$12x^2 - 24x + x - 2 = 0$$
$$12x(x - 2) + 1(x - 2) = 0$$
$$(12x + 1)(x - 2) = 0$$
$$x - 2 = 0 \quad \text{gives} \quad x = 2$$
and
$$12x + 1 = 0 \quad \text{gives,} \quad x = -\frac{1}{12}$$

Number is 2 or $-\frac{1}{12}$

5. The difference of a number and its reciprocal is $\frac{15}{4}$. Find the number.

Solution: Let the number be x.

Then,
$$x - \frac{1}{x} = \frac{15}{4}$$
$$4x^2 - 4 = 15x \text{ (Multiplying by } 4x)$$
$$4x^2 - 15x - 4 = 0$$
$$4x^2 - 16x + x - 4 = 0$$
$$4x(x - 4) + 1(x - 4) = 0$$
$$(x - 4)(4x + 1) = 0$$
$$x - 4 = 0 \quad \text{gives} \quad x = 4$$
and
$$4x + 1 = 0 \quad \text{gives} \quad x = -\frac{1}{4}$$

Number is 4 and $-\frac{1}{4}$

6. The sum of the squares of two digits of a positive integral number is 65 and the number is 9 times the sum of its digits. Find the number.

Solution: Let xy be the number, where unit digit is y and tens digit is x.

According the given condition.

$$x^2 + y^2 = 65 \quad \ldots\ldots\ldots (A)$$

$$\text{Number} = y + 10x$$

$$y + 10x = 9(x + y)$$

$$y + 10x = 9x + 9y$$

$$10x - 9x = 9y - y$$

$$x = 8y \quad \ldots\ldots\ldots (B)$$

Putting $x = 8y$ in (A)

$$(8y)^2 + y^2 = 65$$

$$64y^2 + y^2 = 65$$

$$65y^2 = 65$$

$$y^2 = 1$$

$$\therefore \; y = \pm 1$$

$$y = 1 \quad \text{(taking +ve value)}$$

Put $y = 1$ in (B)

$$x = 8(1) = 8$$

Therefore, number is 8, 1

7. ***The sum of the co-ordinates of a point is 9 and sum of their squares is 45. Find the co-ordinates of the point.***

Solution: Let $P(x, y)$ be the point.

According to the given conditions.

$$x + y = 9 \quad \ldots\ldots\ldots (A)$$

and

$$x^2 + y^2 = 45 \quad \ldots\ldots\ldots (B)$$

From A $\quad x = 9 - y \ldots\ldots\ldots (C)$

Putting $x = 9 - y$ in (B)

$$(9 - y)^2 + y^2 = 45$$

$$81 - 18y + y^2 + y^2 = 45$$

$$2y^2 - 18y + 81 - 45 = 0$$

$$2y^2 - 18y + 36 = 0$$

$$y^2 - 9y + 18 = 0 \quad \text{(Dividing by 2)}$$

$$y^2 - 6y - 3y + 18 = 0$$

$$y(y - 6) - 3(y - 6) = 0$$

$$(y - 6)(y - 3) = 0$$

$$y - 6 = 0 \quad \text{gives} \quad y = 6$$

Then from (C)

$$x = 9 - 6 = 3$$

Point is $P(3, 6)$

When $y - 3 = 0$ then $y = 3$

From (C)

$$x = 9 - 3 = 6$$

Point is (6, 3)

8. ***Find two integers whose sum is 9 and the difference of their squares is also 9.***

Solution: Let the integers be x, y.

Then, $x + y = 9$ .... (A)

and $x^2 - y^2 = 9$ .... (B)

From A

$$x = 9 - y$$

Putting $x = 9 - y$ in (B)

$$(9 - y)^2 - y^2 = 9$$
$$81 - 18y + y^2 - y^2 = 9$$
$$-18y = 9 - 81$$
$$-18y = -72$$
$$y = -\frac{72}{18}$$
$$y = 4$$

Putting $y = 4$ in (A)

$$x + 4 = 9$$
$$x = 9 - 4 = 5$$

Integers are 5, 4

9. ***Find two integers whose difference is 4 and whose squares differ by 72.***

Solution: Let the integers by x and.

according to the given conditions.

$x - y = 4$ .... (A)

and $\quad x^2 - y^2 = 72 \quad \ldots\ldots\ldots (B)$

$x - y = 4 \quad$ from (A)

$\therefore \; x = 4 + y$

Putting $\quad x = 4 + y$ in (B), we get

$$(4 + y)^2 - y^2 = 72$$
$$16 + y^2 + 8y - y^2 = 72$$
$$8y = 72 - 16$$
$$8y = 56$$
$$y = \frac{56}{8}$$
$$y = 7$$

Put $\quad y = 7$ in (A)

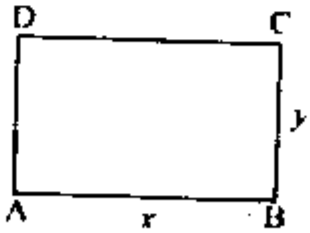
$$x - 7 = 4$$
$$x = 4 + 7 = 11$$

Integers are: 11 and 7

10. ***Find the dimensions of a rectangle, whose perimeter is 80cm and its area is 375cm².***

Solution:

Let x and y be the length and width respectively of the rectangle.



According to the given conditions.

Perimeter: $\quad 2x + 2y = 80$

or $\quad x + y = 40 \quad \ldots\ldots\ldots\ldots (A)$

Area: $\quad xy = 375 \quad \ldots\ldots\ldots\ldots (B)$

$x = 40 - y$ (from A)

Putting $\quad x = 40 - y$ in (B)

$$(40 - y)y = 375$$
$$40y - y^2 = 375$$

$$\Rightarrow \quad y^2 - 40y + 375 = 0$$
$$y^2 - 25y - 15y + 375 = 0$$
$$y(y - 25) - 15(y - 25) = 0$$
$$(y - 15)(y - 25) = 0$$
$$y - 15 = 0 \quad \text{gives} \quad y = 15$$
Putting $y = 15$ in (A)
$$x + 15 = 40$$
$$x = 40 - 15 = 25$$
Length = 25cm, Breadth = 15 cm.

MISCELLANEOUS EXERCISE – 2

I. ***Multiple Choice Questions***

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) If α, β are the roots of $3x^2 + 5x - 2 = 0$, then $\alpha + \beta$ is

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$

(c) $\frac{-5}{3}$ (d) $\frac{-2}{3}$

(ii) If α, β are the roots of $7x^2 - x + 4 = 0$, then $\alpha\beta$ is

(a) $\frac{-1}{7}$ (b) $\frac{4}{7}$

(c) $\frac{7}{4}$ (d) $\frac{-4}{7}$

(iii) Roots of the equation $4x^2 - 5x + 2 = 0$ are

(a) irrational (b) imaginary

(c) rational (d) none of these

(iv) Cube roots of -1 are

(a) $-1, -\omega, -\omega^2$ (b) $-1, \omega, -\omega^2$

3 VARIATIONS

unit, students will learn how to

- define ratio, proportions and variations (direct and inverse)
- find 3^{rd}, 4^{th}, mean and continued proportion.
- apply theorem of invertendo, alternendo, componendo, dividend and componendo & dividend to find proportions.
- define joint variation.
- solve problems related to joint variation.
- use k-method to prove conditional equalities involving proportions.
- solve real life problems based on variations.

EXERCISE 3.1

1. Express the following as a ratio a : b and as a fraction in its simplest (lowest) form.

(i) Rs. 750, Rs. 1250

Solution: Ratio of Rs. 750 to Rs. 1250.

$$750 : 1250$$

$$3 : 5 \text{ and in fraction form } \frac{3}{5}$$

(ii) 450cm, 3m

Solution: Ratio of 450cm to 300cm.

$$450 : 300 \qquad \text{Dividing by 150}$$

$3 : 2$ and in fraction form $= \frac{3}{2}$

(iii) 4kg, 2kg 750 gm

Solution: 2kg 750gm = 2000 + 750gm
2750 gm
Ratio of 4kg to 2kg 750gm
$4000 : 2750$ Dividing by 250
$16 : 11$ and in fraction form $= \frac{16}{11}$

(iv) 27min. 30 sec, 1 hour

Solution: 27min 30 sec , 1 hour
$27 \times 60 + 30$ sec , $1 \times 60 \times 60$ sec
$1620 + 30$
1650 sec , 3600 sec
Ratio is $1650 : 3600$
$1650 : 3600$
$165 : 360$
$33 : 72$
$11 : 24$ or $\frac{11}{24}$

(v) 75^{O}, 225^{O}

Solution:
Ratio is $75 : 225$ Dividing by 75
$1 : 3$ or $\frac{1}{3}$

2. In a class of 60 students, 25 students are girls and remaining students are boy. Compute the ratio of

(i) boys to total students

Solution:
Number of boys $60 - 25$
$= 35$
Total number of students 60

Ratio of boys to total students

$$35 : 60 \qquad \text{Dividing by 5}$$
$$7 : 12$$

(ii) ***boys to girls***

Solution:

Number of boys = 35
Number of girls = 25

Ratio boy to girls

$$35 : 25 \qquad \text{Dividing by 5}$$
$$7 : 5$$

3. ***If $3(4x - 5y) = 2x - 7y$, find the ratio $x : y$.***

Solution:

$$3(4x - 5y) = 2x - 7y$$
$$12x - 15y = 2x - 7y$$
$$12x - 2x = 15y - 7y$$
$$10x = 8y$$
$$\frac{10x}{y} = \frac{8y}{y}$$
$$\frac{10x}{y} = 8$$
$$\frac{10}{10} \times \frac{x}{y} = \frac{8}{10}$$
$$\frac{x}{y} = \frac{4}{5}$$

Ratio $x : y$ is $4 : 5$

4. ***Find the value of p, if the ratios $2p + 5 : 3p + 4$ and $3 : 4$ are equal.***

Solution: $2p + 5 : 3p + 4 = 3 : 4$

$$\Rightarrow \frac{2p + 5}{3p + 4} = \frac{3}{4}$$

$$\Rightarrow \quad 4(2p+5) = 3(3p+4)$$
$$8p + 20 = 9p + 12$$
$$8p - 9p = 12 - 20$$
$$-p = -8$$
Thus. $p = 8$

5. ***If the ratios 3x + 1 : 6 + 4x and 2 : 5 are equal. Find the value of x.***

Solution: $3x+1 : 6+4x = 2 : 5$

$$\Rightarrow \quad \frac{3x+1}{6+4x} = \frac{2}{5}$$
$$\Rightarrow \quad 5(3x+1) = 2(6+4x)$$
$$15x + 5 = 12 + 8x$$
$$15x - 8x = 12 - 5$$
$$7x = 7$$
$$x = \frac{7}{7}$$
$$x = 1$$

6. ***Two numbers are in the ratio 5 : 8. If 9 is added to each number, we get a new ratio 8 : 11. Find the numbers.***

Solution: Let the numbers be $5x$, $8x$

Now $$\frac{5x+9}{8x+9} = \frac{8}{11}$$
$$11(5x+9) = 8(8x+9)$$
$$55x + 99 = 64x + 72$$
$$55x - 64x = 72 - 99$$
$$-9x = -27$$
$$x = \frac{-27}{-9}$$
$$x = 3$$
Numbers are: $5x$, $8x$

$$5(3), (8)(3)$$
$$15, 24$$

7. ***If 10 is added in each number of the ratio 4 : 13, we get a new ratio 1 : 2. What are numbers?***

Solution: Let the numbers be $4x$, $13x$

$$\text{Now, } \frac{4x+10}{13x+10} = \frac{1}{2}$$
$$2(4x+10) = 1(13x+10)$$
$$8x+20 = 13x+10$$
$$8x-13x = 10-20$$
$$-5x = -10$$
$$x = 2$$

Numbers are: $4x, 13x$
$$(4)(2), 13(2)$$
$$8, 26$$

8. ***Find the cost of 8kg of mangoes, if 5kg of mangoes cost Rs. 250.***

Solution: Let Rs. x be the cost of 8kg of mangoes.

Weight of mangoes = Price of mangoes

$$8 : 5 :: x : 250$$

Product of means = Product of extremes

$$5x = 8 \times 250$$
$$x = \frac{8 \times 250}{5}$$
$$= 8 \times 50$$
$$= \text{Rs. } 400$$

9. ***If $a : b = 7 : 6$, find the value of $3a + 5b : 7b - 5a$.***

Solution:

$$a : b = 7 : 6$$
$$\Rightarrow \frac{a}{b} = \frac{7}{6}$$

Now $3a + 5b : 7b - 5a$ Dividing by b

$$\frac{3a}{b} + \frac{5b}{b} : \frac{7b}{b} - \frac{5a}{b}$$

$$3\left(\frac{a}{b}\right) + 5 : 7 - 5\left(\frac{a}{b}\right)$$

Putting $\frac{a}{b} = \frac{7}{6}$ in it.

$$3\left(\frac{7}{6}\right) + 5 : 7 - 5\left(\frac{7}{6}\right)$$

$$\frac{17}{2} : \frac{7}{6}$$

$$\frac{7}{2} + 5 : 7 - \frac{35}{6}$$

$$\frac{7+10}{2} : \frac{42-35}{6}$$

$$\frac{17}{6} : \frac{7}{6}$$

Multiplying by 6

$$6\left(\frac{17}{2}\right) : 6\left(\frac{7}{6}\right)$$

$$51 : 7$$

Thus, $3a + 5b : 7b - 5a = 51 : 7$

10. ***Complete the following:***

(i) If $\frac{24}{7} = \frac{6}{x}$, then $4x =$ ___7___

Here, $24x = 6 \times 7$

$$\frac{24x}{6} = \frac{6 \times 7}{6}$$

$$4x = 7$$

(ii) ***If*** $\frac{5a}{3x} = \frac{15b}{y}$, ***then*** $ay = \underline{\quad 9bx \quad}$

$$5ay = (15b)(3x)$$

$$ay = \frac{(15b)(3x)}{5} = 9bx$$

(iii) ***If*** $\frac{9pq}{2lm} = \frac{18p}{5m}$, ***then*** $5q = \underline{\quad 4l \quad}$

$$\frac{9pq}{2lm} = \frac{18p}{5m} \quad \text{Dividing by } 9p$$

$$\frac{q}{2lm} = \frac{2}{5m}$$

Now $\quad q = \frac{2(2lm)}{5m}$

$$q = \frac{4l}{5}$$

Thus, $\quad 5q = 4l$

11. ***Find x in the following proportions.***

(i) $3x - 2 : 4 :: 2x + 3 : 7$

Solution: $3x - 2 : 4 :: 2x + 3 : 7$

$$\Rightarrow \quad \frac{3x-2}{4} = \frac{2x+3}{7}$$

or $\quad 7(3x - 2) = 4(2x + 3)$

$$21x - 14 = 8x + 12$$

$$21x - 8x = 12 + 14$$

$$13x = 26$$

$$x = \frac{26}{13}$$

$$x = 2$$

(ii) $\frac{3x-1}{7} : \frac{3}{5} :: \frac{2x}{3} : \frac{7}{5}$

Solution: Product of ext = Product of means

$$\frac{7}{5}\left(\frac{3x-1}{7}\right) = \left(\frac{3}{5}\right)\left(\frac{2x}{3}\right)$$

$$\frac{3x-1}{5} = \frac{2x}{5}$$

$$\Rightarrow \quad 3x - 1 = 2x$$

$$3x - 2x = 1$$

$$x = 1$$

(iii) $\frac{x-3}{2} : \frac{5}{x-1} :: \frac{x-1}{3} : \frac{4}{x+4}$

Solution: Product of ext = Product of means

$$\left(\frac{x-3}{2}\right)\left(\frac{4}{x+4}\right) = \left(\frac{5}{x-1}\right)\left(\frac{x-1}{3}\right)$$

$$\frac{2(x-3)}{x+4} = \frac{5}{3}$$

$$2 \times 3(x-3) = 5(x+4)$$

$$6x - 18 = 5x + 20$$

$$6x - 5x = 20 + 18$$

$$x = 38$$

(iv) $p^2 + pq + q^2 : x :: \frac{p^3 - q^3}{p+q} : (p-q)^2$

Solution:

or $x : p^2 + pq + q^2 :: (p-q)^2 : \frac{p^3 - q^3}{p+q}$

$$\Rightarrow \frac{x}{p^2 + pq + q^2} = \frac{(p-q)^2}{p^3 - q^3} \times (p+q) \qquad \text{N.T.S.}$$

$$x = \frac{(p-q)^2(p+q)}{p^3 - q^3} \times (p^2 + pq + q^2)$$

$$= \frac{(p-q)(p-q)(p+q)}{(p-q)(p^2 + pq + q^2)} \times (p^2+p+q^2)$$

$$= (p - q)(p + q)$$
$$x = p^2 - q^2$$

(v) $8 - x : 11 - x :: 16 - x : 25 - x$

Solution: Product of ext = Product of means

$$(8 - x)(25 - x) = (11 - x)(16 - x)$$
$$200 - 33x + x^2 = 176 - 27x + x^2$$
$$x^2 - 33x - x^2 + 27x = 176 - 200$$
$$-6x = -24$$
$$x = \frac{-24}{-6}$$
$$x = 4$$

EXERCISE 3.2

1. If y varies directly as x, and y = 8 when x = 2, find

(i) y in x terms of x ***(ii) y when x = 5***

(iii) x when y = 28

Solution: y varies directly as x

Thus. $y \propto x$

$y = kx$

Where k is constant of variation.

(i) y in terms of x

Put $y = 8, x = 2$ in

$$8 = k(2)$$
$$2k = 8$$
$$k = 4$$

Thus. $y = kx$ becomes

$y = 4x$ (as $k = 4$) (A)

(ii) To find y when $x = 5$

Now $y = 4x$ (from A)

Put $x = 5$ in it

$y = 4 \times 5$

$y = 20$

(iii) ***To find x when*** $y = 28$

Put $y = 28$ in (A)

$y = 4x$ in it

$28 = 4x$

$x = \frac{28}{4}$

$x = 7$

2. ***If*** $y \propto x$, ***and*** $y = 7$ ***when*** $x = 3$ ***find***

(i) y ***in terms of*** x

(ii) x ***when*** $y = 35$ ***and*** y ***when*** $x = 18$

Solution: $y \propto x$

Thus, $y = kx$ $\quad (A)$

Putting $y = 7$ and $x = 3$

Then, $7 = k(3)$

$k = \frac{7}{3}$

(A) becomes

(i) $y = \frac{7}{3}x$ $\quad (B)$

(ii) $y = \frac{7}{3}x$

Put $y = 35$ in it

$35 = \frac{7}{3}x$

$x = \frac{35 \times 3}{7}$

Thus, $x = 15$

Now put $x = 18$ in B

$$y = \frac{7}{3}x$$

$$y = \frac{7}{3}(18)$$

$$y = 7 \times 6$$

$$y = 42$$

3. ***If $R \propto T$ and $R = 5$ when $T = 8$, find the equation connecting R and T. Also find R when $T = 64$ and T when $R = 20$.***

Solution: $R \propto T$

Thus, $R = kT$ (A)

Put $R = 5$ and $T = 8$ in it.

$$5 = k(8)$$

$$k = \frac{5}{8}$$

Put $k = \frac{5}{8}$ in (A)

$$R = \frac{5}{8}T \qquad (B)$$

(ii) Put $T = 64$ in (B)

$$R = \frac{5}{8}(64)$$

$$R = 5 \times 8 = 40$$

Now put $R = 20$ in (B)

$$R = \frac{5}{8}T$$

$$20 = \frac{5}{8}T$$

$$T = \frac{20 \times 8}{5}$$

$$= 4 \times 8$$

$T = 32$

4. ***If $R \propto T^2$ and $R = 8$ when $T = 3$, find R when $T = 6$.***

Solution: $R \propto T^2$

Thus, $R = kT^2$ (A)

Put $R = 8$, $T = 3$ in it.

$8 = k(3)^2$

$8 = 9k$

$k = \frac{8}{9}$

Put $k = \frac{8}{9}$ in A

$R = \frac{8}{9}T^2$ ... (B)

Now put $T = 6$ in (B)

$R = \frac{8}{9}(6)^2$

$= \frac{8}{9} \times 36$

$= 8 \times 4$

$R = 32$

5. ***If $V \propto R^3$ and $V = 5$ when $R = 3$, find R, when $V = 625$.***

Solution: $V \propto R^3$

Thus, $V = kR^3$ (A)

Put $V = 5$, $R = 3$ in it.

$5 = k(3)^3$

$8 = k(27)$

$k = \frac{5}{27}$

Put $k = \frac{5}{27}$ in A

$$V=\frac{5}{27}R^3 \ldots (B)$$

Now put $V=625$ in (B)

$$625=\frac{5}{27}R^3$$

$$R^3=\frac{625\times 27}{5}$$

$$=5^3\times 3^3$$

$$R^3=(15)^3$$

Thus $R=15$

6. ***If w varies directly as u^3 and $w=81$ when $u=3$. Find w, when $u=5$.***

Solution: $w \propto u^3$

Thus, $w=ku^3$ (A)

Put $w=81$ and $u=3$ in (A)

$$81=k(3)^3$$

$$81=27k$$

$$k=\frac{81}{27}=3$$

Put $k=3$ in A

$$w=3u^3 \quad \ldots(B)$$

Now put $u=5$ in (B)

$$w=3(5)^3$$

$$w=3\times 125$$

Thus, $w=375$

7. ***If y varies inversely as x and $y=7$ when $x=2$, find y when $x=126$.***

Solution: $y\propto \frac{1}{x}$

$$y=\frac{k}{x}$$

$yx = k$ (A)

Put $y = 7$ and $x = 2$ in (A)

$(7)(2) = k$

$k = 14$

Putting $k = 14$ in A

$yx = 14$ (B)

Now put $x = 126$ in B,

$y(126) = 14$

$\therefore y = \frac{14}{126}$

$y = \frac{1}{9}$

8. ***If $y \propto \frac{1}{x}$ and $y = 4$ when $x = 3$, find x when $y = 24$.***

Solution: $y \propto \frac{1}{x}$

$y = \frac{k}{x}$

$yx = k$ (A)

Put $y = 4$, $x = 3$ in (A), we get

$(4)(3) = k$

$k = 12$

Putting $k = 12$ in A

$yx = 12$ (B)

Now put $y = 24$ in B,

$24x = 12$

$x = \frac{12}{24}$

$x = \frac{1}{2}$

9. If $w \propto \frac{1}{z}$ and $w = 5$ when $z = 7$, find w when $z = \frac{175}{4}$.

Solution: $w \propto \frac{1}{z}$

Thus, $w = \frac{k}{z}$

$wz = k$ (A)

Put $w = 5$, $z = 7$ in (A)

$(5)(7) = k$

$k = 35$

Putting $k = 35$ in A

$wz = 35$ (B)

Now, put $z = \frac{175}{4}$ in B,

$w\frac{175}{4} = 35$

$w = \frac{4 \times 35}{175}$

Therefore, $w = \frac{4}{5}$

10. $A \propto \frac{1}{r^2}$ and $A = 2$ when $r = 3$, find r when $A = 72$.

Solution: $A \propto \frac{1}{r^2}$

Thus, $A = \frac{k}{r^2}$

$Ar^2 = k$ (A)

Put $A = 2$ and $r = 3$ in (A)

$2(3)^2 = k$

$18 = k$

Putting $k = 18$ in A

$Ar^2 = 18$ (B)

Now, put $A = 72$ in B,

$$72r^2 = 18$$

$$r^2 = \frac{18}{72}$$

$$r^2 = \frac{1}{4}$$

$$r = \pm\frac{1}{2}$$

11. $a \propto \frac{1}{b^2}$ ***and a = 3 when b = 4, find a, when b = 8.***

Solution: $a \propto \frac{1}{b^2}$

$$\therefore \quad a = \frac{k}{b^2}$$

$ab^2 = k$ (A)

Put $a = 3, b = 4$ in (A)

$$3(4)^2 = k$$

$$3 \times 16 = k$$

$k = 48$ in A

Put $k = 48$ (B)

$ab^2 = 48$ in B,

Now, put $b = 8$ in B

$$a(8)^2 = 48$$

$$a \times 64 = 48$$

$$a = \frac{48}{64}$$

$$a = \frac{3}{4}$$

12. $V \propto \frac{1}{r^3}$ ***and $V = 5$ when $r = 3$, find V when $r = 6$ and r when $V = 320$.***

Solution: $V \propto \frac{1}{r^3}$

$\therefore \quad V = \frac{k}{r^3}$

$\Rightarrow \quad Vr^3 = k \qquad$ (A)

Put $V = 5, r = 3$ in (A)

$5(3)^3 = k$

$5 \times 27 = k$

$k = 135$

Put $k = 135$ in (A)

$Vr^3 = 135$ in B,

(i) Put $r = 6$ in B

$V(6)^3 = 135$

$216V = 135$

$V = \frac{135}{216} = \frac{5}{8}$

(ii) Put $V = 320$ in B

$320r^3 = 135$

$r^3 = \frac{135}{320} = \frac{27}{64}$

$r^3 = \left(\frac{3}{4}\right)^3$

$\therefore \; r = \frac{3}{4}$

13. $m \propto \frac{1}{n^3}$ ***and $m = 2$ when $n = 4$, find m when $n = 6$ and n when $m = 432$.***

Solution: $m \propto \dfrac{1}{n^3}$

$$\therefore \quad m = \frac{k}{n^3}$$

$$mn^3 = k \qquad \text{(A)}$$

Put $m = 2,\ n = 4$ in (A)

$$(2)(4)^3 = k$$

$$128 = k$$

Put $k = 128$ in (A)

$$mn^3 = 128 \qquad \text{in B}$$

(i) Now, Put $n = 6$ in B

$$m(6)^3 = 128$$

$$216m = 128$$

$$m = \frac{128}{216} = \frac{16}{27}$$

(ii) Put $m = 432$ in B

$$432n^3 = 128$$

$$n^3 = \frac{128}{432} = \frac{8}{27} = \left(\frac{2}{3}\right)^3$$

$$\therefore \ n = \frac{2}{3}$$

In $a : b :: b : c$

c is called the third proportional and b is called the mean proportional.

In $a : b :: c : d$

d is called the fourth proportional.

EXERCISE 3.3

1. Find a third proportional to

(i) 6, 12

Solution: Let x be the third proportional,

Then $6 : 12 :: 12 : x$

Product of extremes = Product meAns.

Thus $6x = 12 \times 12$

$$x = \frac{12 \times 12}{6}$$

$$x = 24$$

(ii) $a^3, 3a^2$

Solution: Let x be the third proportional,

Then $a^3 : 3a^2 :: 3a^2 : x$

Thus $a^3x = 3a^2 \times 3a^2$

$$x = \frac{3a^2 \times 3a^2}{a^3}$$

$$= 9a$$

(iii) $a^2 - b^2, a - b$

Solution: Let x be the third proportional, then

$(a^2 - b^2) : (a - b) :: (a - b) : x$

Product of extremes = Product meAns.

Thus $x(a^2 - b^2) = (a - b)(a - b)$

$$x = \frac{(a-b)(a-b)}{a^2 - b^2}$$

$$= \frac{(a-b)(a-b)}{(a-b)(a+b)}$$

$$x = \frac{a-b}{a+b}$$

(iv) $(x - y)^2, x^3 - y^3$

Solution: Let 'z' be the third proportional,

Then $(x-y)^2 : (x^3-y^3) :: (x^3-y^3) : x$

Thus $(x-y)^2 z = x^3 - y^3(x^3-y^3)$

$$z = \frac{(x^3-y^3)(x^3-y^3)}{(x-y)^2}$$

$$= \frac{\cancel{(x-y)}(x^2+xy+y^2)\cancel{(x-y)}(x^2+xy+y^2)}{\cancel{(x-y)}\cancel{(x-y)}}$$

$$= (x^2+xy+y^2)^2$$

(v) $(x+y)^2, x^2-xy-2y^2$

Solution: Let z be the third proportional, then

$(x+y)^2 : (x^2-xy-2y^2) :: (x^2-xy-2y^2) : z$

Product of extreme = Product me**Ans.**

Thus $z(x+y)^2 = (x^2-xy-2y^2)(x^2-xy-2y^2)$

$$z = \frac{(x^2-xy-2y^2)(x^2-xy-2y^2)}{(x+y)^2}$$

$$= \frac{[x^2-2xy+xy-2y^2][x^2-2xy+xy-2y^2]}{(x+y)^2}$$

$$= \frac{[x(x-2y)+y(x-2y)][x(x-2y)+y(x-2y)]}{(x+y)^2}$$

$$= \frac{(x+y)(x-2y)(x+y)(x-2y)}{(x+y)(x+y)}$$

$$z = (x-2y)^2$$

(vi) $\dfrac{p^2-q^2}{p^3+q^3}, \dfrac{p-q}{p^2-pq+q^2}$

Solution: Let r be the third proportional, then

$$\frac{p^2-q^2}{p^3+q^3} : \frac{p-q}{p^2-pq+q^2} :: \frac{p-q}{p^2-pq+q^2} : r$$

Product of extreme = Product me**Ans.**

$$\left(\frac{p^2-q^2}{p^3+q^3}\right)(r) = \frac{p-q}{p^2-pq+q^2}\times\frac{p-q}{p^2-pq+q^2}$$

$$\therefore\ r = \frac{p-q}{p^2-pq+q^2}\times\frac{p-q}{p^2-pq+q^2}\times\frac{p^3+q^3}{p^2-q^2}$$

$$= \frac{(p-q)\cancel{(p-q)}\cancel{(p+q)}\cancel{(p^2-pq+q^2)}}{(p^2-pq+q^2)\cancel{(p^2-pq+q^2)}\cancel{(p-q)}\cancel{(p+q)}}$$

$$r = \frac{p-q}{p^2-pq+q^2}$$

2. ***Find a fourth proportional to***

(i) **5, 8, 15**

Solution: Let x be the fourth proportional, then

$$5:8 :: 15:x$$

Product of extremes = Product meAns.

$$5x = 8\times 15$$

$$x = \frac{8\times 15}{5}$$

$$x = 8\times 3$$

$$x = 24$$

(ii) ***$4x^4, 2x^3, 18x^5$***

Solution: Let y be the fourth proportional, then

$$4x^4 : 2x^3 :: 18x^5 : y$$

Product of extremes = Product meAns.

$$(4x^4)y = (2x^3)(18x^5)$$

$$y = \frac{(2x^3)(18x^5)}{4x^4}$$

$$y = 9x^4$$

(iii) ***$15a^5b^6, 10a^2b^5, 21a^3b^3$***

Solution: Let x be the fourth proportional, then

$$15a^5b^6 : 10a^2b^5 :: 21a^3b^3 : x$$

Product of extremes = Product meAns.

$$(15a^5b^6)x = (10a^2b^5)(21a^3b^3)$$
$$x = \frac{210a^5b^8}{15a^5b^6}$$
$$x = 14b^2$$

(iv) $x^2 - 11x + 24, (x-3), 5x^4 - 40x^3$

Solution: Let y be the fourth proportional, then

$(x^2 - 11x + 24) : x - 3 \quad :: \ (5x^4 - 40x^3) : y$

Product of extremes = Product meAns.

$$(x^2 - 11x + 24)y = (x-3)(5x^4 - 40x^3)$$
$$\therefore \qquad y = \frac{(x-3)(5x^4 - 40x^3)}{x^2 - 11x + 24}$$
$$= \frac{(x-3)(5x^3)(x-8)}{x^2 - 8x - 3x + 24}$$
$$= \frac{5x^3(x-3)(x-8)}{x(x-8) - 3(x-8)}$$
$$= \frac{5x^3(x-3)(x-8)}{(x-3)(x-8)}$$
$$x = 5x^3$$

(v) $p^3 + q^3, p^2 - q^2, p^2 - pq + q^2$

Solution: Let x be the fourth proportional, then

$(p^3 + q^3) : p^2 - q^2 :: (p^4 - pq + q^2) : x$

Product of extremes = Product meAns.

$$(p^3 + q^3)x = (p^2 - q^2)(p^2 - pq + q^2)$$
$$\therefore \qquad x = \frac{(p^2 - q^2)(p^2 - pq + q^2)}{p^3 + q^3}$$
$$= \frac{(p+q)(p-q)(p^2 - pq + q^2)}{(p+q)(p^2 - pq + q^2)}$$
$$x = p - q$$

(vi) $(p^2 - q^2)(p^2 + pq + q^2), p^3 + q^3, p^3 - q^3$

Solution: Let x be the fourth proportional, then

$(p^2-q^2)(p^2+pq+q^2):(p^3+q^3)::(p^3-q^3):x$

Product of extremes = Product meAns.

$$(p^2-q^2)(p^2+pq+q^2)x=(p^3+q^3)(p^3-q^3)$$

$$x=\frac{(p^3+q^3)(p^3-q^3)}{(p^2-q^2)(p^2+pq+q^2)}$$

$$=\frac{(p+q)(p^2-pq+q^2)(p-q)(p^2+pq+q^2)}{(p+q)(p-q)(p^2+pq+q^2)}$$

$$x=p^2-pq+q^2$$

3. *Find a mean proportional between*

(i) 20, 45

Solution: Let x be the mean proportional, then

$$20:x::x:45$$

Product of means = **Product extremes**

$$x^2=20\times 45$$
$$x^2=900$$
$$x=\pm 30$$

(ii) $20x^3y^5$, $5x^7y$

Solution: Let z be the mean proportional, then

$$20x^3y^5:z::z:5x^7y$$

Product of means = Product extremes

$$z^2=20x^3y^5\times 5x^7y$$
$$=100x^{10}y^6$$
$$z^2=(10x^5y^3)^2$$

Thus, $z=\pm 10x^5y^3$

(iii) $15p^4qr^3$, $135q^5r^7$

Solution: Let z be the mean proportional, then

$$15p^4qr^3:z::z:135q^5r^7$$

Product of means = Product extremes.

$$z^2=(15p^4qr^3)(135)q^5r^7$$
$$=2025p^4q^6r^{10}$$
$$z^2=(45)^2(p^2q^3r^5)^2$$

$$\therefore \quad z = \pm 45p^2q^3r^5$$

(iv) $x^2 - y^2, \frac{x-y}{x+y}$

Solution: Let z be the mean proportional, then

$$(x^2 - y^2) : z :: z : \frac{x-y}{x+y}$$

Product of means = Product extremes

$$z^2 = (x^2 - y^2)\left(\frac{x-y}{x+y}\right)$$

$$= \frac{(x+y)(x-y)(x-y)}{(x+y)}$$

$$z^2 = (x-y)^2$$

Thus, $z = \pm(x - y)$

4. *Find the values of the letter involved in the following continued proportions.*

(i) ***5, p, 45***

Solution: Here, $5 : p :: p : 45$

Product of means = Product extremes

$$p \times p = 5 \times 45$$

$$p^2 = 225$$

$$p^2 = (15)^2$$

Thus, $p = \pm 15$

(ii) ***8, x, 18***

Solution: Here, $8 : x :: x : 18$

Product of means = Product extremes

$$x \times x = 8 \times 18$$

$$x^2 = 144$$

$$x^2 = (12)^2$$

Thus, $x = \pm 12$

(iii) $\quad 12,\ 3p-6,\ 27$

Solution:

Here, $12 : (3p-6) :: (3p-6) : 27$

Product of means = Product extremes

$$(3p-6)(3p-6) = 12\times 27$$
$$9p^2 - 36p + 36 = 12\times 27$$
$$9p^2 - 36p + 36 = 324$$
$$9p^2 - 36p + 36 - 324 = 0$$
$$9p^2 - 36p - 288 = 0 \qquad \text{Dividing by 9}$$
$$p^2 - 4p - 32 = 0$$
$$p^2 - 8p + 4p - 32 = 0$$
$$p(p-8) + 4(p-8) = 0$$
$$(p-8)(p+4) = 0$$
$$p - 8 = 0 \qquad \text{gives} \quad p = 8$$
$$p + 4 = 0 \qquad \text{gives} \quad p = -4$$

(iv) $\quad 7,\ m-3,\ 28$

Solution:

Here, $7 : (m-3) :: (m-3) : 28$

Product of means = Product extremes

$$(m-3)(m-3) = 7\times 28$$
$$m^2 - 6m + 9 = 196$$
$$m^2 - 6m + 9 - 196 = 0$$
$$m^2 - 6m - 187 = 0$$
$$m^2 - 17m + 11m - 187 = 0$$
$$m(m-17) + 11(m-17) = 0$$
$$(m-17)(m+11) = 0$$
$$m - 17 = 0 \qquad \text{gives} \quad m = 17$$
$$m + 11 = 0 \qquad \text{gives} \quad m = -11$$

Theorems on Proportions

(i) Invertendo

If $\frac{a}{b} = \frac{c}{d}$ then $\frac{b}{a} = \frac{d}{c}$

(ii) Alternando

If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{c} = \frac{b}{d}$

(iii) Componendo

If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{b} = \frac{c+d}{d}$

$\frac{a}{a+b} = \frac{c}{c+d}$

(iv) Dividendo

If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a-b}{b} = \frac{c-d}{d}$

and $\frac{a}{a-b} = \frac{c}{c-d}$

(v) Componendo-dividendo.

If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

EXERCISE 3.4

1. ***Prove that*** $a : b = c : d$, ***if***

(i) $$\frac{4a+5b}{4a-5b} = \frac{4c+5d}{4c-5d}$$

By componendo-dividendo

$$\frac{(4a+5b)+(4a-5b)}{(4a+5b)-(4a-5b)} = \frac{(4c+5d)+(4c-5d)}{(4c+5d)-(4c-5d)}$$

$$\frac{4a+5b+4a-5b}{4a+5b-4a+5b}=\frac{4c+5d+4c-5d}{4c+5d-4c+5d}$$

$$\frac{8a}{10b}=\frac{8c}{10d}$$

$$\frac{a}{b}=\frac{c}{d}$$

Thus, $a:b=c:d$ (proved)

(ii) $$\frac{2a+9b}{2a-9b}=\frac{2c+9d}{2c-9d}$$

By componendo-dividendo

$$\frac{(2a+9b)+(2a-9b)}{(2a+9b)-(2a-9b)}=\frac{(2c+9d)+(2c-9d)}{(2c+9d)-(2c-9d)}$$

$$\frac{2a+9b+2a-9b}{2a+9b-2a+9b}=\frac{2c+9d+2c-9d}{2c+9d-2c+9d}$$

$$\frac{4a}{18b}=\frac{4c}{18d}$$

Thus, $a:b=c:d$ (proved)

(iii) $$\frac{ac^2+bd^2}{ac^2-bd^2}=\frac{c^3+d^3}{c^3-d^3}$$

By componendo-dividendo

$$\frac{(ac^2+bd^2)+(ac^2-bd^2)}{(ac^2+bd^2)-(ac^2-bd^2)}=\frac{(c^3+d^3)+(c^3-d^3)}{(c^3+d^3)-(c^3-d^3)}$$

$$\frac{ac^2+bd^2+ac^2-bd^2}{ac^2+bd^2-ac^2+bd^2}=\frac{c^3+d^3+c^3-d^3}{c^3+d^3-c^3+d^3}$$

$$\frac{2ac^2}{2bd^2}=\frac{2c^3}{2d^3}\quad \text{(Multiply by } \frac{d^2}{c^2}\text{)}$$

$$\frac{a}{b}=\frac{c}{d}$$

Thus, $a:b=c:d$ (proved)

(iv) $$\frac{a^2c+b^2d}{a^2c-b^2d}=\frac{ac^2+bd^2}{ac^2-bd^2}$$

By componendo-dividendo

$$\frac{(a^2c+b^2d)+(a^2c-b^2d)}{(a^2c+b^2d)-(a^2c-b^2d)}=\frac{(ac^2+bd^2)+(ac^2-bd^2)}{(ac^2+bd^2)-(ac^2-bd^2)}$$

$$\frac{a^2c+b^2d+a^2c-b^2d}{a^2c+b^2d-a^2c+b^2d}=\frac{ac^2+bd^2+ac^2-bd^2}{ac^2+bd^2-ac^2+bd^2}$$

$$\frac{2a^2c}{2b^2d}=\frac{2ac^2}{2bd^2}$$

$$\frac{a^2c}{b^2d}=\frac{ac^2}{bd^2}$$

$$\frac{a\times ac}{b\times bd}=\frac{(ac)c}{(bd)d}$$

$$\frac{a}{b}=\frac{c}{d}$$

Thus, $a:b=c:d$ (proved)

(v) $pa+qb:pa-qb=pc+qd:pc-qd$

Solution: $$\frac{pa+qb}{pa-qb}=\frac{pc+qd}{pc-qd}$$

By componendo-dividendo

$$\frac{(pa+qb)+(pa-qb)}{(pa+qb)-(pa-qb)}=\frac{(pa+qb)+(pa-qb)}{(pa+qb)-(pa-qb)}$$

$$\frac{pa+qb+pa-qb}{pa+qb-pa+qb}=\frac{pc+qd+pc-qd}{pc+qd-pc+qd}$$

$$\frac{2pa}{2qb}=\frac{2pc}{2qd}$$

$$\frac{pa}{qb}=\frac{pc}{qd}$$

$$\frac{a}{b}=\frac{c}{d}$$

Thus, $a:b=c:d$ (proved)

(vi) $$\frac{a+b+c+d}{a+b-c-d}=\frac{a-b+c-d}{a-b-c+d}$$

By componendo-dividendo

$$\frac{(a+b+c+d)+(a+b-c-d)}{(a+b+c+d)-(a+b-c-d)}=\frac{(a-b+c-d)+(a-b-c+d)}{(a-b+c-d)-(a-b-c+d)}$$

$$\frac{a+b+c+d+a+b-c-d}{a+b+c+d-a-b+c+d}=\frac{a-b+c-d+a-b-c+d}{a-b+c-d-a+b+c-d}$$

$$\frac{2a+2b}{2c+2d}=\frac{2a-2b}{2c-2d}$$

$$\frac{a+b}{c+d}=\frac{a-b}{c-d}$$

By alternando

$$\frac{a+b}{c-b}=\frac{c+d}{c-d}$$

$$\frac{a+b+a-b}{a+b-a+b}=\frac{c+d+c-d}{c+d-c+d}$$

$$\frac{2a}{2b}=\frac{2c}{2d}$$

$$\frac{a}{b}=\frac{c}{d}$$

Thus, $a:b=c:d$ (proved)

(vii) $$\frac{2a+3b+2c+3d}{2a+3b-2c-3d}=\frac{2a-3b+2c-3d}{2a-3b-2c+3d}$$

By componendo-dividendo

$$\frac{2a+3b+2c+3d+2a+3b-2c-3d}{2a+3b+2c+3d-2a-3b+2c+3d}=\frac{2a-3b+2c-3d+2a-3b-2c+3d}{2a-3b+2c-3d-2a+3b+2c-3d}$$

$$\frac{4a+6b}{4c+6d}=\frac{4a-6b}{4c-6d}$$

$$\frac{2(2a+3b)}{2(2c+3d)}=\frac{2(2a-3b)}{2(2c-3d)}$$

$$\frac{2a+3b}{2c+3d}=\frac{2a-3b}{2c-3d}$$

By alternando

$$\frac{2a+3b}{2a-3b}=\frac{2c+3d}{2c-3d}$$

By componendo-dividendo

$$\frac{2a+3b+2a-3b}{2a+3b-2a+3b}=\frac{2c+3d+2c-3d}{2c+3d-2c+3d}$$

$$\frac{4a}{6b}=\frac{4c}{6d}$$

$$\frac{a}{b}=\frac{c}{d}$$

Thus, $a:b=c:d$ (proved)

(viii) $$\frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$$

By componendo-dividendo

$$\frac{a^2+b^2+a^2-b^2}{a^2+b^2-a^2+b^2}=\frac{ac+bd+ac-bd}{ac+bd-ac+bd}$$

$$\frac{2a^2}{2b^2}=\frac{2ac}{2bd}$$

$$\frac{a^2}{b^2}=\frac{ac}{bd}$$

$$\frac{a\times a}{b\times b}=\frac{a\times c}{b\times d}$$

$$\frac{a}{b}=\frac{c}{d}$$

Thus. $a:b=c:d$ (proved)

2. ***Using theorem of componendo-dividendo***

(i) ***Find the value of*** $\frac{x+2y}{x-2y}+\frac{x+2z}{x-2z}$, ***if*** $x=\frac{4yz}{y+z}$

Solution:

$$x=\frac{4yz}{y+z}$$

$$x=\frac{2y(2z)}{y+z}$$

$$\frac{x}{2y}=\frac{2z}{y+z}$$

using componendo-dividendo

$$\frac{x+2y}{x-2y}=\frac{2z+y+z}{2z-y-z}$$

$$\frac{x+2y}{x-2y}=\frac{3z+y}{z-y} \qquad (A)$$

Now

$$x=\frac{4zy}{y+z}$$

$$x=\frac{2z(2y)}{y+z}$$

$$\frac{x}{2z}=\frac{2y}{y+z}$$

Applying componendo-dividendo.

$$\frac{x+2z}{x-2z}=\frac{2y+y+z}{2y-y-z}$$

$$\frac{x+2y}{x-2y}=\frac{3y+z}{y-z}$$

From A + B

$$\frac{x+2y}{x-2y}+\frac{x+2y}{x-2y}=\frac{3z+y}{z-y}+\frac{3y+z}{y-z}$$

$$=\frac{3z+y}{z-y}-\frac{3y+z}{z-y}$$

$$=\frac{3z+y-3y-z}{z-y}$$

$$=\frac{2z-2y}{z-y}$$

$$=\frac{2(z-y)}{(z-y)}$$

$$=2 \quad \text{Answer.}$$

(ii) ***Find the value of*** $\frac{m+5n}{m-5n}+\frac{m+5p}{m-5p}$, ***if*** $m=\frac{10np}{n+p}$

Solution:

$$m=\frac{10np}{n+p}$$

$$m=\frac{(5n)(2p)}{n+p}$$

$$\frac{m}{5n}=\frac{2p}{n+p}$$

By componendo-dividendo

$$\frac{m+5n}{m-5n}=\frac{2p+n+p}{2p-n-p}$$

$$=\frac{3p+n}{p-n} \qquad \text{(A)}$$

Now $$m=\frac{10np}{n+p} \qquad \text{(given)}$$

$$m=\frac{(2n)(5p)}{n+p}$$

$$\frac{m}{5p} = \frac{2n}{n+p}$$

By componendo-dividendo

$$\frac{m+5p}{m-5p} = \frac{2n+n+p}{2n-n-p}$$

$$= \frac{3n+p}{n-p} \qquad \text{(B)}$$

From (A) + (B)

$$\frac{m+5n}{m-5n} + \frac{m+5P}{m-5P} = \frac{3p+n}{p-n} + \frac{3n+P}{n-P}$$

$$= \frac{3p+n}{p-n} - \frac{3n+p}{p-n}$$

$$= \frac{3p+n-3n-p}{p-n}$$

$$= \frac{2p-2n}{p-n}$$

$$\frac{m+5n}{m-5n} + \frac{m+5p}{m-5p} = \frac{2(p-n)}{(p-n)}$$

$$= 2$$

(iii) Find the value of $\frac{x-6a}{x+6a} - \frac{x+6b}{x-6b}$***, if*** $x = \frac{12ab}{a-b}$

Solution:

$$x = \frac{12ab}{a-b}$$

$$x = \frac{(6a)(2b)}{a-b}$$

$$\frac{x}{6a} = \frac{2b}{a-b}$$

Apply dividend-componendo.

$$\frac{x-6a}{x+6a}=\frac{2b-a+b}{2b+a-b}$$

$$=\frac{3b-a}{b+a} \qquad \text{(A)}$$

Now $$x=\frac{12ab}{a-b}$$

$$x=\frac{(2a)(6b)}{a-b}$$

$$\frac{x}{6b}=\frac{2a}{a-b}$$

Apply componendo-dividendo.

$$\frac{x+6b}{x-6b}=\frac{2a+a-b}{2a-a+b}$$

$$=\frac{3a-b}{a+b} \qquad \text{(B)}$$

From (A) – (B)

$$\frac{x-6a}{x+6a}-\frac{x+6b}{x-6b}=\frac{3b-a}{b+a}-\frac{3a-b}{a+b}$$

$$=\frac{3b-a-3a+b}{a+b}$$

$$=\frac{2b-4a}{a+b}$$

$$\frac{x-6a}{x+6a}-\frac{x-6b}{x+6b}=\frac{2(b-2a)}{a+b}$$

(iv) ***Find the value of*** $\frac{x-3y}{x+3y}-\frac{x+3z}{x-3z}$, *if* $x=\frac{3yz}{y-z}$

Solution: $$x=\frac{3yz}{y-z}$$

$$\frac{x}{3y} = \frac{z}{y-z}$$

By dividend-componendo

$$\frac{x-3y}{x+3y} = \frac{z-y+z}{z+y-z}$$

$$= \frac{2z-y}{y} \qquad \textbf{(A)}$$

$$x = \frac{3yz}{y-z}$$

$$\frac{x}{3z} = \frac{y}{y-z}$$

By componendo-dividendo.

$$\frac{x+3z}{x-3y} = \frac{y+y-z}{y-y+z}$$

$$= \frac{2y-z}{z} \qquad \text{(B)}$$

From (A) (B)

$$\frac{x-3y}{x+3y} - \frac{x+3z}{x-3z} = \frac{2z-y}{y} - \frac{2y-z}{z}$$

$$= \frac{z(2z-y)-y(2y-z)}{yz}$$

$$= \frac{2z^2-yz-2y^2+yz}{yz}$$

$$= \frac{2z^2-2y^2}{yz}$$

$$= \frac{2(z^2-y^2)}{yz}$$

(v) ***Find the value of*** $\dfrac{s-3p}{s+3p}+\dfrac{s+3q}{s-3q}$, ***if*** $s=\dfrac{6pq}{P-q}$

Solution:

$$s=\frac{6pq}{p-q}$$

$$s=\frac{(3p)(2q)}{p-q}$$

$$\frac{s}{3p}=\frac{2q}{p-q}$$

By dividend-componendo

$$\frac{s-3p}{s+3p}=\frac{2q-p+q}{2q+p-q}$$

$$=\frac{3q-p}{q+p} \qquad \text{(A)}$$

$$s=\frac{6pq}{p-q} \qquad \text{(given)}$$

$$\therefore \quad \frac{s}{3q}=\frac{2p}{p-q}$$

By componendo-dividendo.

$$\frac{s+3q}{s-3q}=\frac{2p+p-q}{2p-p+q}$$

$$=\frac{3p-q}{p+q} \qquad \text{(A)}$$

From (A + B)

$$\frac{s-3p}{s+3p}+\frac{s+3p}{s-3p}=\frac{3q-p}{q+p}+\frac{3p-q}{p+q}$$

$$=\frac{3q-p+3p-q}{p+q}$$

$$=\frac{2p+2q}{p+q}$$

$$= \frac{2(p+q)}{(p+q)}$$

$$= 2$$

(vi) ***Solve*** $\frac{(x-2)^2-(x-4)^2}{(x-2)^2+(x-4)^2} = \frac{12}{13}$

Solution: Apply componendo-dividendo

$$\frac{(x-2)^2-(x-4)^2+(x-2)^2+(x-4)^2}{(x-2)^2-(x-4)^2-(x-2)^2-(x-4)^2} = \frac{12+13}{12-13}$$

$$\frac{2(x-2)^2}{-2(x-4)^2} = \frac{25}{-1}$$

$$\frac{(x-2)^2}{(x-4)^2} = 25$$

Taking sq. root.

$$\frac{x-2}{x-4} = \pm 5$$

When $\frac{x-2}{x-4} = 5$	When $\frac{x-2}{x-4} = -5$
$x-2 = 5x-20$	$x-2 = -5x+20$
$x-5x = -20+2$	$x+5x = 20+2$
$-4x = -18$	$6x = 22$
$x = \frac{-18}{-4}$	$x = \frac{22}{6}$
$= \frac{9}{2}$	$x = \frac{11}{3}$

(vii) *Solve* $\dfrac{\sqrt{x^2+2}+\sqrt{x^2-2}}{\sqrt{x^2+2}-\sqrt{x^2-2}}=2$

Solution: Apply componendo-divid...

$$\frac{\sqrt{x^2+2}+\sqrt{x^2-2}+\sqrt{x^2+2}-\sqrt{x^2-2}}{\sqrt{x^2+2}+\sqrt{x^2-2}-\sqrt{x^2+2}+\sqrt{x^2-2}}=\frac{2+1}{2-1}$$

$$\frac{2\sqrt{x^2+2}}{2\sqrt{x^2-2}}=\frac{3}{1}$$

$$\frac{\sqrt{x^2+2}}{\sqrt{x^2-2}}=3$$

$$\frac{x^2+2}{x^2-2}=(3)^2 \quad \text{(squaring)}$$

$$x^2+2=9(x^2-2)$$

$$x^2+2=9x^2-18$$

$$x^2-9x^2=-18-2$$

$$-8x^2=-20$$

$$x^2=-\frac{20}{-8}$$

$$x^2=\frac{5}{2}$$

$$x=\pm\sqrt{\frac{5}{2}}$$

(viii) *Solve* $\dfrac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}}=\dfrac{1}{3}$

Apply componendo-dividendo

$$\frac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}+\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}-\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}}=\frac{1+3}{1-3}$$

$$\frac{2\sqrt{x^2+8p^2}}{-2\sqrt{x^2-p^2}}=\frac{4}{-2}$$

$$\frac{\sqrt{x^2+8p^2}}{\sqrt{x^2-p^2}} = +2 \quad \text{squaring}$$

$$\frac{x^2+8p^2}{x^2-p^2} = 4$$

$$x^2+8p^2 = 4(x^2-p^2)$$

$$x^2+8p^2 = 4x^2-4p^2$$

$$x^2-4x^2 = -4p^2-8p^2$$

$$-3x^2 = -12p^2$$

$$x^2 = 4p^2 \quad \text{(Taking sq. root)}$$

$$x = \pm 2p$$

(ix) *Solve* $\dfrac{(x+5)^3-(x-3)^3}{(x+5)^3+(x-3)^3} = \dfrac{13}{14}$

Solution: Apply componendo-dividendo

$$\frac{(x+5)^3-(x-3)^3+(x+5)^3+(x-3)^3}{(x+5)^3-(x-3)^3-(x+5)^3-(x-3)^3} = \frac{13+14}{13-14}$$

$$\frac{2(x+5)^3}{-2(x-3)^3} = \frac{27}{-1}$$

$$\frac{(x+5)^3}{(x-3)^3} = 27$$

$$\left(\frac{x+5}{x-3}\right)^3 = (3)^3$$

$$\therefore \quad \frac{x+5}{x-3} = 3$$

$$x+5 = 3(x-3)$$

$$x+5 = 3x-9$$

$$x-3x = -9-5$$

$$-2x = -14$$

$$x = \frac{-14}{-2}$$

$$x = 7$$

EXERCISE 3.5

1. If s varies directly as u^2 and inversely as v and s = 7 when u = 3, v = 2. Find the value of s when u = 6 and v = 10.

Solution: $s \propto \frac{u^2}{v}$

$\therefore \quad s = k\frac{u^2}{v} \quad (A)$

Put $s = 7, u = 3, v = 2$ in (A)

$7 = k\frac{(3)^2}{2}$

$7 = \frac{9k}{2}$

$\frac{7\times 2}{9} = k$

$k = \frac{14}{9}$

Put $k = \frac{14}{9}$ in(A)

$s = \frac{14}{9}\times\frac{u^2}{v}$ .... (B)

Now Put $u = 6, v = 10$ in (B)

$s = \frac{14}{9}\times\frac{(6)^2}{10}$

$$= \frac{\overset{7}{\cancel{14}}}{\underset{1}{\cancel{9}}} \times \frac{\overset{4}{\cancel{36}}}{\underset{5}{\cancel{10}}}$$

$$s = \frac{28}{5}$$

2. ***If w varies jointly as x, y^2 and z and w = 5 when x = 2, y = 3, z = 10. Find w when x = 4, y = 7 and z = 3.***

Solution: $W \propto xy^2z$

$\therefore \quad W = kxy^2z$ .... (A)

Put $W = 5, x = 2, y = 3, z = 10$ in (A)

$5 = k(2)(3)^2(10)$

$5 = 180k$

$k = \frac{5}{180} = \frac{1}{36}$

Put $k = \frac{1}{36}$ in (A)

$W = \frac{1}{36}xyz$..........(B)

Put $x = 4, y = 7, z = 3$ in (B)

$W = \frac{1}{36}(4)(7)^2(3)$

$= \frac{1}{36} \times 4 \times 49 \times 3$

$W = \frac{49}{3}$

3. ***If y varies directly as x^3 and inversely as z^2 and t, and $y = 16$ when $x = 4$, $z = 2$, $t = 3$. Find the value of y when $x = 2$, $z = 3$ and $t = 4$.***

Solution: $y \propto \dfrac{x^3}{z^2 t}$

$$y = \frac{kx^3}{z^2 t} \text{ .... (A)}$$

Put $y = 16$, $x = 4$, $z = 2$, $t = 3$ in (A)

$$16 = \frac{k(4)^3}{(2)^2(3)}$$

$$16 = \frac{k64}{4\times 3} = \frac{16k}{3}$$

$$k = 16 \times \frac{3}{16} = 3$$

Put $k = 3$ in (A)

$$y = \frac{3x^3}{z^2 t} \text{ ....(B)}$$

Put $x = 2$, $z = 3$, $t = 4$ in (B)

$$y = \frac{3(2)^3}{(3)^2(4)}$$

$$= \frac{\overset{1}{\cancel{2}} \times \overset{4}{\cancel{8}}}{9 \times \underset{2}{\cancel{4}}}$$

$$y = \frac{4}{9}$$

4. ***If u varies directly as x^2 and inversely as the product yz^3, and $u = 2$ when $x = 8$, $y = 7$, $z = 2$. Find the value of u when $x = 6$, $y = 3$, $z = 2$.***

Solution: $u \propto \dfrac{x^2}{yz^3}$

$$u = \frac{kx^2}{yz^3} \text{..............} (A)$$

Put $u = 2, x = 8, y = 7, z = 2$ in (A)

$$2 = \frac{k(8)^2}{7(2)^3}$$

$$2 = \frac{k\times 8\times 8}{7\times 8} = \frac{k\times 8}{7}$$

$$k = \frac{2\times 7}{8}$$

$$k = \frac{7}{4}$$

Put $k = \dfrac{7}{4}$ in A.

$$u = \frac{7x^2}{4yz^3} \text{...(B)}$$

Now, Put $x = 6, y = 3, z = 2$ in

$$u = \frac{7(6)^2}{4(3)(2)^3}$$

$$= \frac{7\times 6\times 6}{4\times 3\times 8}$$

$u = \frac{21}{8}$

5. ***If v varies directly as the product xy^3 and inversely as z^2 and $v = 27$ when $x = 7$, $y = 6$, $z = 7$. Find the value of v when $x = 6$, $y = 2$, $z = 3$.***

Solution: $v \propto \frac{xy^3}{z^2}$

$v = k\frac{xy^3}{z^2}$ ....(A)

Put $v = 27,\ x = 7,\ y = 6,\ z = 7$

$27 = \frac{k(7)(6)^3}{(7)^2}$

$27 = \frac{k(7)(6)(6)(6)}{7\times 7}$

$k = \frac{27\times 7}{6\times 6\times 6}$

$= \frac{7}{8}$

Put $k = \frac{7}{8}$ in(A)

$v = \frac{7}{8}\times\frac{xy^3}{z^2}$ ..... (B)

Put $x = 6,\ y = 2,\ z = 3$

$v = \frac{7}{8}\times 6\times\frac{(2)^3}{3^2}$

$$\frac{7}{8}\times 6\times\frac{8}{9^3}$$

$$v=\frac{14}{3}$$

6. ***If w varies inversely as the cube of u, and w = 5 when u = 3. Find w, when u = 6.***

Solution: $w\propto\frac{1}{u^3}$

$$w=\frac{k}{u^3}\ \ldots\ldots(A)$$

Put $w=5, u=3$ in (A)

$$5=\frac{k}{(3)^3}$$

$$5=\frac{k}{27}$$

$$k=5\times 27$$

$$=135$$

Put $k=135$ in (A)

$$w=\frac{135}{u^3}\ \ldots\ldots\ldots(B)$$

Put $u=6$ in (B)

$$w=\frac{135}{(6)^3}$$

$$= \frac{135}{6\times 6\times 6}$$

$$w = \frac{5}{8}$$

K-method

EXERCISE 3.6

1. If a : b = c : d, (a, b, c, d ≠ 0), then show that

(i) $\frac{4a-9b}{4a+9b} = \frac{4c-9d}{4c+9d}$

Solution: $a : b = c : d$

$\Rightarrow \frac{a}{b} = \frac{c}{d}$

Let $\frac{a}{b} = \frac{c}{d} = k$

Thus. $a = bk$, $c = dk$

L.H.S.	R.H.S.
$\frac{4a-9b}{4a+9b}$ (i)	$\frac{4c-9d}{4c+9d}$ (ii)
Put $a = bk$ in (i)	Put $c = dk$ in (ii)
$\frac{4bk-9b}{4bk+9b}$	$\frac{4dk-9d}{4dk+9d}$
$= \frac{b(4k-9)}{b(4k+9)}$	$= \frac{d(4k-9)}{d(4k+9)}$
$= \frac{4k-9}{4k+9}$	$= \frac{4k-9}{4k+9}$

$\therefore$ L.H.S $=$ R.H.S.

(ii) *If* $\frac{a}{b} = \frac{c}{d}$ *then* $\frac{6a-5b}{6a+5b} = \frac{4c-5d}{4c+5d}$

Let $\frac{a}{b} = \frac{c}{d} = k$

Thus, $a = bk$

$c = dk$

L.H.S.	R.H.S.
$\frac{6a-5b}{6a+5b}$	$\frac{6c-5d}{6c+5d}$
Put $a = kk$	Put $c = kd$
$\frac{6bk-5b}{6bk+5b}$	$\frac{6dk-5d}{6dk+5d}$
$= \frac{b(6k-5)}{b(6k+5)}$	$= \frac{d(6k-5)}{d(6k+5)}$
$= \frac{6k-5}{6k+5}$	$= \frac{6k-5}{6k+5}$

L.H.S. = R.H.S.

(iii) *If* $\frac{a}{b} = \frac{c}{d}$ *then* $\frac{a}{b} = \sqrt{\frac{a^2+c^2}{b^2+d^2}}$

Let $\frac{a}{b} = \frac{c}{d} = k$

Then $a = bk$

$c = dk$

L.H.S.	R.H.S.
$\frac{a}{b}$ (i)	$\sqrt{\frac{a^2+c^2}{b^2+d^2}}$ (ii)
Put $a = bk$ in (i)	Put $a = bk$,

$\frac{bk}{b}$	$c = dk$ in (ii)
	$= \sqrt{\frac{(bk)^2 + (dk)^2}{b^2 + d^2}}$
k	$= \sqrt{\frac{b^2k^2 + d^2k^2}{(b^2 + d^2)}}$
	$= \sqrt{\frac{k^2(b^2 + d^2)}{(b^2 + d^2)}}$
	$= \sqrt{k^2}$
	$= k$

L.H.S. = R.H.S.

Thus, $\frac{a}{b} \quad \sqrt{\frac{a^2 + c^2}{b^2 + d^2}}$

(iv) ***If*** $\frac{a}{b} = \frac{c}{d}$ ***then*** $a^6 + c^6 : b^6 + d^6 = a^3c^3 : b^3d^3$

Let $\frac{a}{b} = \frac{c}{d} \quad k$

$a \quad bk, c \quad dk$

L.H.S.	R.H.S.
$\frac{a^6 + c^6}{b^6 + d^6}$ .... (i)	$\frac{a^3c^3}{b^3d^3}$ .... (ii)
Put $a \quad bk$	Put $c = dk$
$c \quad dk$, in (i)	$a = bk$ in (ii)
$\frac{(bk)^6 + (dk)^6}{b^6 + d^6}$	$\frac{(bk)^3(dk)^3}{b^3d^3}$

$$=\frac{b^6k^6+d^6k^6}{b^6+d^6}$$

$$=\frac{k^6(b^6+d^6)}{b^6+d^6}$$

$$=k^6$$

$$=\frac{b^3k^3d^3k^3}{b^3d^3}$$

$$=k^{3+3}$$

$$=k^6$$

L.H.S. = R.H.S.

Thus, $a^6+c^6:b^6+d^6=a^3c^3:b^3d^3$

(v) ***If*** $\frac{a}{b}=\frac{c}{d}$ ***then*** $p(a+b)+qb:p(c+d)+qd=a:c$

Let $\frac{a}{b}=\frac{c}{d}=k$

Thus, $a=bk$

$c=dk$

L.H.S.

$$=\frac{p(a+b)+qb}{p(c+d)+qd} \quad (i)$$

Put $a=bk$

$c=dk$ in (i)

$$\frac{p(bk+b)+qb}{p(dk+d)+qd}$$

$$\frac{pb(k+1)+qb}{pd(k+1)+qd}$$

$$=\frac{b}{d}\frac{[(k+1)+q]}{[p(k+1)+q]}$$

$$=\frac{b}{d}$$

R.H.S.

$$=\frac{a}{c} \;.... (ii)$$

Put $c=dk$

$a=bk$ in (ii)

$$\frac{bk}{dk}$$

$$=\frac{b}{d}$$

L.H.S. = R.H.S.

(vi) *If* $\frac{a}{b}=\frac{c}{d}$ *then* $a^2+b^2:\frac{a^3}{a+b}=c^2+d^2:\frac{c^3}{c+d}$

Let $\frac{a}{b}=\frac{c}{d}=k$ then $a=bk$

$c=dk$

L.H.S.	R.H.S.
$\frac{a^2+b^2}{\frac{a^3}{a+b}}$	$\frac{c^2+d^2}{\frac{c^3}{c+d}}$
$=(a^2+b^2)\times\frac{(a+b)}{a^3}$ (i)	$=(c^2+d^2)\times\frac{(c+d)}{c^3}$ (ii)
Put $a=bk$ in (i)	Put $c=dk$ in (ii)
$\left[(bk)^2+b^2\right]\times\left[\frac{bk+b}{b^3k^3}\right]$	$=(d^2k^2+d^2)\times\left(\frac{dk+d}{b^3k^3}\right)$
$=(b^2k^2+b^2)\left(\frac{bk+b}{b^3k^3}\right)$	$=\frac{d^2(k^2+1)(d)(k+1)}{d^3k^3}$
$=\frac{b^2(k^2+1)(b)(k+1)}{b^3k^3}$	
$\frac{(k^2+1)(k+1)}{k^3}$	$\frac{(k^2+1)(k+1)}{k^3}$

L.H.S. = R.H.S.

(vii) *If* $\frac{a}{b}=\frac{c}{d}$ *then* $\frac{a}{a-b}:P=\frac{800\times120^{12}}{200\times40_{10}}=\frac{c}{c-d}:\frac{c+d}{d}$

Let $\frac{a}{b}=\frac{c}{d}=k$ $\Rightarrow$ $\frac{a}{a-b}\times\frac{b}{a+b}=\frac{c}{c-d}\times\frac{d}{c+d}$

then $a=bk$

$c=dk$

L.H.S.	R.H.S.
$\frac{a}{a-b}\times\frac{b}{a+b}$	$\frac{c}{c-d}\times\frac{d}{c+d}$
Put $a = bk$	Put $c = dk$
$\frac{bk}{bk-b}\times\frac{b}{bk+b}$	$\frac{dk}{dk-d}\times\frac{d}{dk+d}$
$=\frac{bk}{b(k-1)}\times\frac{b}{b(k+1)}$	$\frac{dk}{d(k-1)}\times\frac{d}{d(k+1)}$
$=\frac{k}{k-1}\times\frac{1}{k+1}$	$\frac{k}{k-1}\times\frac{1}{k+1}$
$=\frac{k}{(k-1)(k+1)}$	$=\frac{k}{(k-1)(k+1)}$

L.H.S. = R.H.S.

EXERCISE 3.7

1. The surface area A of a cube varies directly as the square of the length l of an edge and A = 27 square units when l = 3 units.

Find (i) A when l = 4 units (ii) l when A = 12 sq. units.

Solution: $A \propto l^2$

$A = kl^2$ .... (A)

Put $A = 27$ square units, $l = 3$ units in it.

$27 = k(3)^2$

$27 = 9k$

$k = \frac{27}{9}$

$k = 3$

Putting $k = 3$ in (A)

$A = 3l^2$ .... (B)

(i) *Now, put $l = 4$ in (B)*

$A = 3(4)^2$

$A = 3 \times 16$

$\boxed{A = 48 \text{ square units}}$

(ii) *Now, put $A = 12$ sq. units in (B)*

$12 = 3l^2$

$\frac{12}{3} = l^2$

$l^2 = 4$ $\boxed{l = 2}$ *Ans.*

2. The surface area S of the sphere varies directly as the square of radius r, and $S = 16\pi$ when $r = 2$. Find r when $S = 36\pi$.

Solution: $S \propto r^2$

$S = kr^2$ .... (A)

Put $S = 16\pi$, $r = 2$ in it.

$16\pi = k(2)^2$

$16\pi = 4k$

$k = \frac{16\pi}{4} = 4\pi$

Putting $k = 4\pi$ in (A)

$S = 4\pi r^2$ .... (B)

Now, put $S = 36\pi$ in (B)

$36\pi = 4\pi r^2$

$r^2 = \frac{36\pi}{4\pi}$

$r^2 = 9$

$r = \sqrt{9}$

$r = 3$

3. *In Hook's law the force F applied to stretch a spring varies directly as the amount of elongation S and F = 32lb when S = 1.6 in. Find (i) S when F = 50lb. (ii) F when S = 0.8 in.*

Solution: $F \propto S$

$F = kS$ .... (A)

Put $F = 32, S = 1.6$

$32 = k(1.6)$

$k = \dfrac{32}{1.6}$

$= \dfrac{32 \times 10}{16}$

$= 2 \times 10 = 20$

Put $k = 20$ in (A)

$F = 20\,S$ .... (B)

(i) Put $F = 50$ in (B)

$50 = 20\,S$

$S = \dfrac{50}{20}$

$S = 2.5$ in

(ii) Put $S = 0.8$ in (B)

$F = (20)(0.8)$

$F = 16\ lb$

4. *The intensity I of light from a given source varies inversely as the square of the distance d from it. If the intensity is 20 candlepower at a distance of 12ft form the source, find the intensity at a point 8ft. from the source.*

Solution: $I \propto \dfrac{1}{d^2}$

$I = \dfrac{k}{d^2}$ ......(A)

Put $l = 20$ and $d = 12$ in it.

$$20 = \frac{k}{(12)^2}$$

$$k = 20 \times (12)^2$$

$$= 20 \times 144$$

$$= 2880$$

Put $k = 2880$ in (A)

$$l = \frac{2880}{d^2} \text{ .... (B)}$$

Now put $d = 8$ in it

$$l = \frac{2880}{(8)^2}$$

$$l = \frac{2880}{64}$$

$$l = 45 \text{ cp.}$$

5. ***The pressure P in a body of fluid varies directly as the depth d. If the pressure exerted on the bottom of a tank by a column of fluid 5ft. high is 2.25 lb/sq. in, how deep much the fluid be to exert a pressure of 9lb/sq. in?***

Solution: $p \propto d$

$$p = kd \text{ .... (A)}$$

Put $p = 2.25,\ d = 5$ in it.

$$2.25 = k(5)$$

$$k = \frac{2.25}{5}$$

$$= 0.45$$

Put $k = 0.45$ in (A)

$p = 0.45d$ .... (B)

Now, $p = 9 lb/sq.$ in

Put p = 9 in (B)

$9 = 0.45d$

$$d = \frac{9}{0.45}$$

$d = 20 ft$

6. ***Labour costs c varies jointly as the number of workers n and the average number of days d, if the cost of 800 workers for 13 days is Rs. 286000, then find the labour cost of 600 workers for 18 days.***

Solution: $c \propto n \times d$

$c = knd$ .... (A)

Put $c = 286000$, $n = 800$, d = 13 in (A)

$286000 = k(800)(13)$

$$k = \frac{286000}{800 \times 13}$$

Put value of k in (A)

$$c = \frac{286000}{800 \times 13} nd \quad \text{..........} \ (B)$$

Now, put n = 600, d = 18 in (B)

$$c = \frac{\overset{\overset{275}{\cancel{2750}}}{\cancel{286000}}}{\cancel{800} \times \cancel{13}} \times 6\cancel{00} \times 18$$

= 297000 Rupees.

7. ***The supporting load c of a pillar varies as the fourth power of its diameter d and inversely as the square of its length l. A pillar of diameter 6 inch and of height 30 feet will support a load of 63 tons. How high a 4 inch pillar must be to support a load of 28 tons?***

Solution: $c \propto \frac{(d)^4}{l^2}$

$$c = k\frac{d^4}{l^2}$$

$$k = \frac{cl^2}{d^4} \text{ .... (A)}$$

Put $c = 63, l = 30, d = 6$

$$k = \frac{63\times(30)^2}{6^4}$$

Put this value of k in A

$$\frac{63\times(30)^2}{6^4} = \frac{cl^2}{d^4} \text{........(B)}$$

Now, put c = 28, d = 4 in (B)

$$\frac{63\times(30)^2}{6^4} = \frac{28\times l^2}{4^4}$$

$$l^2 = \frac{63\times(30)^2}{6^4}\times\frac{4^4}{28}$$

$$= \frac{\overset{9}{\cancel{63}}\times\overset{5}{\cancel{30}}\times\overset{5}{\cancel{30}}\times 4\times\overset{2}{\cancel{4}}\times\overset{2}{\cancel{4}}\times\cancel{4}}{\underset{1}{\cancel{6}}\times\underset{1}{\cancel{6}}\times\cancel{6}\times\cancel{6}\times\underset{7}{\cancel{28}}}$$

$$l^2 = 400$$

$$l = \sqrt{400}$$

$$l = 20 \ ft.$$

8. ***The time T required for an elevator to lift a weight varies jointly as the weight w and the lifting depth d varies inversely as the power p of the motor. If 25 sec. are required for a 4-hp motor to lift 500 lb through 40 ft, what power is required to lift 800 lb, through 120 ft in 40 sec.?***

Solution: $T \propto \frac{wd}{p}$

$$T = k \times \frac{wd}{p}$$

$$25 = \frac{k \times 500 \times 400}{4}$$

$$\frac{25 \times 4}{500 \times 400} = k$$

$$k = \frac{1}{200}$$

$$T = k\frac{wd}{p}$$

$$40 = \frac{1}{200} \times \frac{800 \times 120}{p}$$

$$P = \frac{\cancel{800}^{4} \times \cancel{120}^{12}}{\cancel{200} \times \cancel{40}_{10}}$$

$$= 12 \text{ hp}$$

9. ***The kinetic energy (K.E.) of a body varies jointly as the mass "m" of the body and the square of its velocity "v". If the kinetic energy is 4320 ft/lb when the mass is 45 lb and the velocity is 24 ft/sec. Determine the***

kinetic energy of a 3000 lb automobile travelling 44 ft/sec.

Solution: $E \propto mv^2$

$E = kmv^2$ ....(A)

Put $E = 4320, m = 45, v = 24$

$4320 = k \times 45 \times (24)^2$

$4320 = k \times 45 \times 576$

$k = \dfrac{4320}{45 \times 576}$

Put this value of k in A

$E = \dfrac{4320}{45 \times 576} \times mv^2$ ....(B)

Put $m = 3000, v = 44$

$E = \dfrac{4320 \times 3000 \times (44)^2}{45 \times 576}$

$= \dfrac{4320 \times 3000 \times 44 \times 44}{45 \times 576}$

$E = 968000$

MISCELLANEOUS EXERCISE – 3

I. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) In a ratio $a : b$, a is called
- (a) relation (b) antecedent
- (c) consequent (d) None of these

(ii) In a ratio $x : y$, y is called
- (a) relation (b) antecedent
- (c) consequent (d) None of these

(iii) In a proportional $a : b : : c : d$, a and d are called,

(a) means (b) extremes
(c) third proportional
(d) None of these

(iv) In a proportional $a : b :: c : d$, b and c are called
(a) means (b) extremes
(c) fourth proportional
(d) None of these

(v) In continued proportional $a : b = b : c$, $ac = b^2$, b is said to be proportional to a and b.
(a) third (b) fourth
(c) means (d) None of these

(vi). In continued proportion $a : b = b : c$, c is said to be proportional to a and b.
(a) third (b) fourth
(c) means (d) None of these

(vii) Find x in proportion $4 : x :: 5 : 15$
(a) $\frac{75}{4}$ (b) $\frac{4}{3}$
(c) $\frac{3}{4}$ (d) 12

(viii) If $u \propto v^2$, then
(a) $u = v^2$ (b) $u = kv^2$
(c) $uv^2 = k$ (d) $uv^2 = 1$

(ix) If $y^2 \propto \frac{1}{x^3}$, then
(a) $y^2 = \frac{k}{x^3}$ (b) $y^2 = \frac{1}{x^3}$
(c) $y^2 = x^2$ (d) $y^2 = kx^3$

(x) If $\frac{u}{v} = \frac{v}{w} = k$, then

(a) $u = wk^2$ (b) $u = vk^2$

(c) $u = w^2k$ (d) $u = v^2k$

(xi) The third proportional of x^2 and y^2 is

(a) $\frac{y^2}{x^2}$ (b) x^2y^2

(c) $\frac{y^4}{x^2}$ (d) $\frac{y^2}{x^4}$

(xii) The fourth proportional w of $x : y :: v : w$ is

(a) $\frac{xy}{v}$ (b) $\frac{vy}{x}$

(c) xyv (d) $\frac{x}{vy}$

(xiii) If $a : b = x : y$, then alternando property is

(a) $\frac{a}{x} = \frac{b}{y}$ (b) $\frac{a}{b} = \frac{x}{y}$

(c) $\frac{a+b}{b} = \frac{x+y}{y}$ (d) $\frac{a-b}{x} = \frac{x-y}{y}$

(xiv) If $a : b = x : y$, then invertendo property is

(a) $\frac{a}{x} = \frac{b}{y}$ (b) $\frac{a}{a-b} = \frac{x}{x-y}$

(c) $\frac{a+b}{b} = \frac{x+y}{y}$ (d) $\frac{b}{a} = \frac{y}{b}$

(xv) If $\frac{a}{b} = \frac{c}{d}$, then componendo property is

(a) $\frac{a}{a+b} = \frac{c}{c+d}$ (b) $\frac{a}{a-b} = \frac{c}{c-d}$

(c) $\frac{ad}{bc}$ (d) $\frac{a-b}{b}=\frac{c-d}{d}$

Answers:

(i)	b	(ii)	c	(iii)	b	(iv)	a	(v)	c
(vi)	a	(vii)	d	(viii)	b	(ix)	a	(x)	a
(xi)	c	(xii)	b	(xiii)	a	(xiv)	d	(xv)	a

2. ***Write short answers of the following questions.***

(i) Define ratio and give one example.

Ans. A relation between quantities of the same kind is called ratio. Let a, b be any two quantities of the same kind their ratio is written as $a : b$.

(ii) Define proportion.

Ans. A proportion is a statement, which is expressed as an equivalence of two ratios, as $a : b = c : d$

(iii) Define direct variation.

Ans. If two quantities are related in such a way that increases (decreases) in one quantity causes increase (decrease) in the other quantity.

(iv) Define inverse variation.

Ans. If two quantities are related in such a way that when one quantity increases, the other decreases is called inverse variation.

(v) State theorem of componendo-dividendo.

Ans. If $\frac{a}{b}=\frac{c}{d}$

then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (Componendo-dividendo)

(vi) Find x, if $6 : x :: 3 : 5$.

Ans. $6 : x = 3 : 5$

$3x = 6 \times 5$

$$x = \frac{6\times5}{3} = 10$$

(vii) If x and y^2 varies directly, and $x = 27$ when $y = 4$. Find the value of y when $x = 3$.

Ans. $x \propto y^2$

$$x = ky^2$$

$$27 = k(4)^2$$

$$k = \frac{27}{16}$$

$$x = \frac{27}{16}y^2$$

$$3 = \frac{27}{16}y^2$$

$$y^2 = \frac{\cancel{3}\times16}{\cancel{27}_{9}}$$

$$y = \pm\frac{4}{3}$$

(viii) If u and v varies inversely, and $u = 8$, when $v = 3$. Find v when $u = 12$.

Ans. $u \propto \frac{1}{v}$

$$u = \frac{k}{v}$$

$$8 = \frac{k}{3}$$

$$k = 24$$

$$u = \frac{24}{v}$$

$$12 = \frac{24}{v}$$

$$v = \frac{24}{12} = 2$$

(ix) Find the fourth proportional 8, 7, 6.

Ans. Let x be the 4th proportional.

then $8 : 7 :: 6 : x$

$$8x = 6 \times 7$$

$$x = \frac{\overset{3}{\cancel{6}} \times 7}{\underset{4}{\cancel{8}}}$$

$$= \frac{21}{4}$$

(x) Find a mean proportional to 16 and 49.

Ans. Let x be the mean proportional

then $16 : x :: x : 49$

then $x^2 = 16 \times 49$

$$x = (4)^2 \times (7)^2$$

$$x = (4 \times 7)^2$$

Thus, $x = \pm 28$

(xi) Find a third proportional to 28 and 4.

Ans. Let x be the third proportional

$28 : 4 :: 4 : x$

$$28x = 4 \times 4$$

$$x = \frac{4 \times 4}{28}$$

$$x = \frac{4}{7}$$

(xii) If $y \propto \frac{x^2}{z}$ and $y = 28$ when $x = 7$, $z = 2$, then find y.

Ans. $y \propto \frac{x^2}{z}$

$$y = k\frac{x^2}{z}$$

$$28 = k\frac{(7)^2}{2}$$

$$28 = \frac{49k}{2}$$

$$k = \frac{28 \times 2}{49}$$

$$= \frac{8}{7}$$

$$y = \frac{8}{7}\frac{x^2}{z}$$

(xiii) If $z \propto yz$ and $z = 36$ when $x = 2$, $y = 3$, then find z.

Ans. $z \propto xy$

$$z = k\,xy \,\ldots\ldots (A)$$

$$36 = k(2)(3)$$

$k = \dfrac{36}{2\times 3} = 6$

(A) becomes $z = 6xy$

(xiv) If $w \propto \dfrac{1}{v^2}$ and $w = 2$ when $v = 3$, then find w.

Ans. $w \propto \dfrac{1}{v^2}$

$w = \dfrac{k}{v^2}$

$2 = \dfrac{k}{3^2}$

$k = 2\times 3^2$

$= 2\times 9$

$= 18$

Thus. $w = \dfrac{18}{v^2}$

3. Fill in the blanks.

(i) The simplest from of the ratio $\dfrac{(x+y)(x^2+xy+y^2)}{x^3-y^3}$ is ______.

(ii) In a ratio $x : y$; x is called ______.

(iii) In a ratio $a : b$; b is called ______.

(iv) In a proportion $a : b :: x : y$; a and y are called ______.

(v) In a proportion $p : q :: m : n$; q and m are called ______.

(vi) In proportion $7 : 4 :: p : 8$, p = ______.

(vii) If $6 : m :: 9 : 12$, then m = ______.

(viii) If x and y varies directly, then x = ______.

(ix) If v varies inversely as u^3, then u^3 = ______.

(x) If w varies inversely as p^2, then k = ______.

(xi) A third proportional of 12 and 4, is ______.

(xii) The fourth proportional of 15, 6, 5 is ______.

(xiii) The mean proportional of $4m^2n^4$ and p^6 is ______

(xiv) The continued proportion of 4, m and 9 is ______

Answers:

(i)	$\frac{x+y}{x-y}$	(ii)	antecedent
(iii)	consequent	(iv)	extremes
(v)	means	(vi)	$p = 14$
(vii)	m 8	(viii)	[illegible]
(ix)	$\frac{v}{k}$	(x)	p^2w
(xi)	$\frac{4}{3}$	(xii)	2
(xiii)	$\pm 2mn^2p^3$	(xiv)	$m = \pm 6$

SUMMARY

- A relation between two quantities of the same kind is called ***ratio***.
- A ***proportion*** is a statement, which is expressed as equivalence of two ratios.
  If two ratios $a : b$ and $c : d$ are equal, then we can write

  $a : b = c : d$

- If two quantities are related in such a way that increase (decrease) in one quantity causes increase (decrease) in the other quantity is called ***direct variation***.

EXERCISE 4.1

Resolve into partial fractions.

1. $\dfrac{7x-9}{(x+1)(x-3)}$

Solution: Let $\dfrac{7x-9}{(x+1)(x-3)} = \dfrac{A}{x+1} + \dfrac{B}{x-3}$ ....(i)

Multiplying by $(x + 1)(x - 3)$, we get

$7x-9 = A(x-3)+B(x+1)$

$7x-9 = Ax-3A+Bx+B$

$7x-9 = Ax+Bx-3A+B$

$7x-9=(A+B)x-3A+B$

Comparing co-efficients of x and constant terms.

$A+B=7$..................(ii)

$-3A+B=-9$..............(iii)

(ii) – (iii) gives.

$4A=16$

$A=4$

Put $A=4$ in (ii)

$4+B=7$

$B=7-4=3$

Putting values of A, B in (i), we get

$$\frac{7x-9}{(x+1)(x-3)} = \frac{4}{x+1} + \frac{3}{(x+3)}$$

2. $\dfrac{x-11}{(x-4)(x+3)}$

Solution: Let $\dfrac{x-11}{(x-4)(x+3)} = \dfrac{A}{x-4} + \dfrac{B}{x-3}$ .... (i)

Multiplying by $(x-4)(x+3)$, we get

$x - 11 = A(x+3) + B(x-4)$

$x - 11 = Ax + 3A + Bx - 4B$

$x - 11 = Ax + Bx + 3A - 4B$

$x - 11 = (A + B)x + (3A - 4B)$

Comparing coefficients of x and constant terms

$A + B = 1$................(ii)

$3A - 4B = -11$.........(iii)

Multiplying (ii) by 3, we get

$3A + 3B = 3$.............(iv)

Subtracting (iii) from (iv)

$7B = 14$

$B = 2$

Put $B = 2$ in (ii)

$A + 2 = 1$

$A = 1 - 2 = -1$

Putting values of A, B in (i), we get

$$\frac{x-11}{(x-4)(x+3)} = \frac{-1}{x-4} + \frac{2}{x+3}$$

3. $\dfrac{3x-1}{x^2-1}$

Solution: Let $\dfrac{3x-1}{x^2-1}=\dfrac{3x-1}{(x-1)(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}$.....$(i)$

Multiplying by $(x-1)(x+1)$, we get

$3x-1=A(x+1)+B(x-1)$

$3x-1=Ax+A+Bx-B$

$3x-1=Ax+Bx+A-B$

$3x-1=(A+B)x+(A-B)$

Comparing coefficients of x and constant term.

$A+B=3$............ **(ii)**

$A-B=-1$.......... **(iii)**

Adding (ii), (iii)

$2A=2$

$A=1$

Put $A=1$ in (ii)

$1+B=3$

$B=3-1=2$

Putting values of A, B in (i), we get

$$\frac{3x-1}{x^2-1}=\frac{1}{x-1}+\frac{2}{x+1}$$

4. $\dfrac{x-5}{x^2+2x-3}$

Solution: $\dfrac{x-5}{x^2+2x-3}$

$$= \frac{x-5}{x^2 - x + 3x - 3}$$

$$= \frac{x-5}{x(x-1)+3(x-1)}$$

$$= \frac{x-5}{(x-1)(x+3)}$$

Let $\frac{x-5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$ .... (i)

Multiplying by $(x-1)(x+3)$, we get

$x-5 = A(x+3) + B(x-1)$

$x-5 = Ax + 3A + Bx - B$

$x-5 = Ax + Bx + 3A - B$

$x-5 = (A+B)x + (3A - B)$

Comparing co-efficients of x and constant terms

$A + B = 1$ ....(ii)

$3A - B = -5$ ....(iii)

$4A = -4$ (from ii + iii)

$A = -1$

Put $A = -1$ in (ii)

$-1 + B = 1$

$B = 1 + 1$

$B = 2$

Putting values of A, B in (i), we get

$$\frac{x-5}{(x-1)(x+3)} = \frac{-1}{x-1} + \frac{2}{x+3}$$

5. $\dfrac{3x+3}{(x-1)(x+2)}$

Solution: Let $\dfrac{3x+3}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$ .....(i)

Multiplying by $(x-1)(x+2)$, we get

$3x + 3 = A(x+2) + B(x-1)$

$3x + 3 = Ax + 2A + Bx - B$

$3x + 3 = Ax + Bx + 2A - B$

$3x + 3 = (A + B)x + (2A - B)$

Comparing coefficients of x and constant terms

$A + B = 3$..........(ii)

$2A - B = 3$........(iii)

$3A = 6$ (from ii + iii)

$A = 2$

Put $A = 2$ in (ii)

$2 + B = 3$

$B = 3 - 2 = 1$

Putting values of A, B in (i), we get

$$\frac{3x+3}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{1}{x+2}$$

6. $\dfrac{7x-25}{(x-4)(x-3)}$

Solution: Let $\dfrac{7x-25}{(x-4)(x-3)} = \dfrac{A}{x-4} + \dfrac{B}{x-3}$ ....(i)

Multiplying by $(x-4)(x-3)$, we get

$7x - 25 = A(x-3) + B(x-4)$

$7x - 25 = Ax - 3A + Bx - 4B$

$7x - 25 = Ax + Bx - 3A - 4B$

$7x - 25 = (A + B)x - 3A - 4B$

Comparing coefficients of x and constant terms.

$A + B = 7$..................(ii)

$-3A - 4B = -25$.........(iii)

Multiply (ii) by 3, we get

$3A + 3B = 21$..............(iv)

$-B = -4$ (from iii + iv)

$B = 4$

Put $B = 4$ in (ii)

$A + 4 = 7$

$A = 7 - 4$

$A = 3$

Putting values of A, B in (i), we get

$$\frac{7x-25}{(x-4)(x-3)} = \frac{3}{x-4} + \frac{4}{x-3}$$

7. $\dfrac{x^2+2x+1}{(x-2)(x+3)}$

Solution: $\dfrac{x^2+2x+1}{(x-2)(x+3)}$

$$= \frac{x^2+2x+1}{x^2+x-6}$$

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2+x-6 & x^2+2x+1 \\ & \pm x^2 \pm x \mp 6 \\ \hline & \quad\quad x+7 \end{array}$$

$$=1+\frac{x+7}{x^2+x-6}$$

$$=1+\frac{x+7}{(x-2)(x+3)}$$

Now, Let $\frac{x+7}{(x-2)(x+3)}=\frac{A}{x-2}+\frac{B}{x+3}$........(*i*)

Multiplying by $(x-2)(x+3)$, we get

$x+7=A(x+3)+B(x-2)$

$x+7=Ax+3A+Bx-2B$

$x+7=Ax+Bx+3A-2B$

$x+7=(A+B)x+(3A-2B)$

Comparing coefficients of x and constant terms.

$A+B=1$...............(ii)

$3A-2B=7$............(iii)

Multiplying (ii) by 2

$2A+2B=2$............(iv)

$5A=9$ (from iii + iv)

$$A=\frac{9}{5}$$

Put $A=\frac{9}{5}$ in (ii)

$$\frac{9}{5}+B=1$$

$$B=1-\frac{9}{5}$$

$$B=\frac{5-9}{5}=-\frac{4}{5}$$

Putting values of A, B in (i), we get

$$\frac{x+7}{(x-2)(x+3)}=\frac{9}{5(x-2)}-\frac{4}{5(x+3)}$$

Hence, $1+\dfrac{x+7}{(x-2)(x+3)}=1+\dfrac{9}{5(x-2)}-\dfrac{4}{5(x+3)}$

8. $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}$

Solution: $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}$

$$=2x+3+\frac{8x-4}{3x^2-2x-1}$$

$$=2x+3+\frac{8x-4}{3x^2-3x+x-1}$$

$$\begin{array}{r|l} & \quad 2x+3 \\ \hline 3x^2-2x-1 & 6x^3+5x^2 \qquad -7 \\ & \underline{\pm 6x^3 \mp 4x^2 \mp 2x} \\ & \qquad 9x^2+2x-7 \\ & \qquad \underline{\pm 9x^2 \mp 6x \mp 3} \\ & \qquad\qquad 8x-4 \end{array}$$

$$=2x+3+\frac{8x-4}{3x(x-1)+1(x-1)}$$

$$=2x+3+\frac{8x-4}{(x-1)(3x+1)}\text{.......}(i)$$

Let $\dfrac{8x-4}{(x-1)(3x+1)}=\dfrac{A}{x-1}+\dfrac{B}{3x+1}$.........(P)

Multiplying by $(x-1)(3x+1)$, we get

$8x-4=A(3x+1)+B(x-1)$

$8x-4=3Ax+A+Bx-B$

$8x-4=3Ax+Bx+A-B$

$8x-4=(3A+B)x+(A-B)$

Comparing coefficients of x and constant terms.

$3A + B = 8$..............(ii)

$A - B = -4$..........(iii)

$4A = 4$ (from ii + iii)

$A = 1$

Put A = 1 in (ii)

$3(1) + B = 8$

$B = 8 - 3$

$B = 5$

Putting values of A, B in, we get

$$\frac{8x-4}{(x-1)(3x+1)} = \frac{1}{x-1} + \frac{5}{3x+1}$$

Hence, $2x+3+\frac{8x-4}{(x-1)(3x+1)} = 2x+3+\frac{1}{x-1}+\frac{5}{3x+1}$

EXERCISE 4.2

Resolve into partial fractions.

1. $\frac{x^2-3x+1}{(x-1)^2(x-2)}$

Solution: Let $\frac{x^2-3x+1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

Multiplying both sides by $(x-1)^2(x-2)$, we get

$x^2-3x+1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$......(i)

$x^2-3x+1 = A(x^2-3x+2) + B(x-2) + C(x^2-2x+1)$

Put $x-1=0$ i.e; $x=1$ in (i)

$(1)^2-3(1)+1=B(1-2)$

$1-3+1=B(-1)$

$-1=-B$

$B=1$

Now, put $x-2=0$ i. e; $x=2$ in (i)

$(2)^2-3(2)+1=C(2-1)^2$

$4-6+1=C(1)^2$

$-1=C(1)$

$C=-1$

Comparing co-efficients of x^2 on both sides

$1=A+C$

Put $C=-1$ in it

$1=A+(-1)$

$1=A-1$

$2=A$

Hence, required partial fractions are

$$\frac{2}{x-1}+\frac{1}{(x-1)^2}-\frac{1}{x-2}$$

Thus, $\dfrac{x^2-3x+1}{(x-1)^2(x-2)}=\dfrac{2}{x-1}+\dfrac{1}{(x-1)^2}-\dfrac{1}{x-2}$

2. $\dfrac{x^2+7x+11}{(x+2)^2(x+3)}$

Solution:

Let $\dfrac{x^2+7x+11}{(x+2)^2(x+3)} = \dfrac{A}{(x+2)} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+3)}$ .... (D)

Multiplying by $(x+2)^2(x+3)$, we get

$x^2+7x+11 = A(x+2)(x+3) + B(x+3) + C(x+2)^2$....(i)

$x^2+7x+11 = A(x^2+5x+6) + B(x+3) + C(x^2+4x+4)$....(ii)

Put $x+2=0$ i.e; $x=-2$ in (i)

$(-2)^2 + 7(-2) + 11 = B(-2+3)$

$4 - 14 + 11 = B(1)$

$1 = B$

$B = 1$

Put $x+3=0$ or $x=-3$ in (i)

$(-3)^2 + 7(-3) + 11 = C(-3+2)^2$

$9 - 21 + 11 = C(-1)^2$

$-1 = C(1)$

$C = -1$

Comparing co-efficients of x^2 in (ii)

$1 = A + C$

Put $C = -1$ in it

$1 = A - 1$

$1 + 1 = A$

$A = 2$

Put values of A, B, C in (D)

Partial fractions are

$$\frac{2}{x+2}+\frac{1}{(x+2)^2}-\frac{1}{x+3}$$

Thus, $\dfrac{x^2+7x+11}{(x+2)^2(x+3)}=\dfrac{2}{x+2}+\dfrac{1}{(x+2)^2}-\dfrac{1}{x+3}$

3. $\dfrac{9}{(x-1)(x+2)^2}$.

Solution: Let $\dfrac{9}{(x+2)^2(x-1)}=\dfrac{A}{x+2}+\dfrac{B}{(x+2)^2}+\dfrac{C}{x-1}$ ..... (L)

Multiplying by $(x+2)^2(x-1)$ we get

$9=A(x+2)(x-1)+B(x-1)+C(x+2)^2$ ....(i)

$9=A(x^2+x-2)+B(x-1)+C(x^2+4x+4)$ ...(ii)

Put $x+2=0$ i.e; $x=-2$ in (i)

$9 = B(-2-1)$

$9 = -3B$

$B = -3$

Put $x-1=0$ i.e; $x=1$ in (i)

$9 = C(1+2)^2$

$9 = C(3)^2$

$9 = C(9)$

$C=1$

Comparing co-efficient of x^2 in (ii)

$0 = A + C$

Put $C = 1$ in it

$0 = A + 1$

$A = -1$

Putting values of A, B, C in (L)

Partial fractions are

$$\frac{-1}{x+2} - \frac{3}{(x+2)^2} + \frac{1}{x-1}$$

$$= \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$$

Thus. $$\frac{9}{(x+2)^2(x-1)} = \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$$

4. $$\frac{x^4+1}{x^2(x-1)}$$

$$= \frac{x^4+1}{x^3-x^2}$$

$$= x+1+\frac{x^2+1}{x^3-x^2}$$

$$= x+1+\frac{x^2+1}{x^2(x-1)} \dots \text{(i)}$$

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2 & x^4 \qquad +1 \\ & \pm x^4 \mp x^3 \\ \hline & x^3 \qquad +1 \\ & \pm x^3 \mp x^2 \\ \hline & x^2+1 \end{array}$$

We take up $\dfrac{x^2+1}{x^2(x-1)}$

Now. Let $$\frac{x^2+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \dots\dots \text{(ii)}$$

Multiplying by $x^2(x-1)$, we get

$$x^2+1 = A(x)(x-1) + B(x-1) + Cx^2 \dots\dots \text{(iii)}$$

Put $x = 0$ in (iii)

$(1)^2 + 1 = B(0-1)$

$1 \quad B(-1)$

$1 \quad -B$

$B = -1$

Put $\quad x - 1 \quad 0$ i.e; $x - 1$ in (iii)

$(i)^2 + 1 \quad C(1)^2$

$1 + 1 \quad C(1)$

$2 \quad C(1)$

$C = 2$

Comparing co-efficients of x^2 in (iii)

$1 \quad A + C$

Put C = 2 in it

$1 \quad A + 2$

$A = 1 - 2 = -1$

Putting values of A, B, C. in (ii) and from (i)

Partial fractions are

$$x + 1 - \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

Thus, $x + 1 + \dfrac{x^2+1}{x^2(x-1)} = x + 1 - \dfrac{1}{x} - \dfrac{1}{x^2} + \dfrac{2}{x-1}$

5. $\quad \dfrac{7x+4}{(3x+2)(x+1)^2}$

Solution:

Let $\quad \dfrac{7x+4}{(x+1)^2(3x+2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{3x+2}$ ....(i)

Multiplying by $(x+1)^2(3x+2)$, we get

$$7x + 4 = A(x+1)(3x+2) + B(3x+2) + C(x+1)^2 \ \ldots(\text{ii})$$

$$7x + 4 = A(3x^2+5x+2) + B(3x+2) + C(x^2+2x+1) \ \ldots\ldots(\text{iii})$$

Put $x + 1 = 0$ i.e; $x = -1$ in (ii)

$$7(-1) + 4 = B\{3(-1) + 2\}$$

$$-7 + 4 = B(-3 + 2)$$

$$-3 = B(-1)$$

$$-3 = -B$$

$$B = 3$$

Put $3x + 2 = 0$ i.e; $x = -\frac{2}{3}$ in (ii)

$$7\left(-\frac{2}{3}\right) + 4 = C\left(-\frac{2}{3} + 1\right)^2$$

$$\frac{-14}{3} + 4 = C\left(\frac{-2+3}{3}\right)^2$$

$$\frac{-14+12}{3} = C\left(\frac{1}{3}\right)^2$$

$$\frac{-2}{3} = \frac{1}{9}C$$

$$C = \left(-\frac{2}{3}\right)(9)$$

$$C = -6$$

Comparing co-efficients of x^2 in (iii)

$$0 = 3A + C$$

Putting $C = -6$ in it

$$0 = 3A + (-6)$$

$0 = 3A - 6$

$3A = 6$

$A = 2$

Putting values of A, B, C in (i)

Partial fractions are

$$\frac{2}{x+1}+\frac{3}{(x+1)^2}-\frac{6}{3x+2}$$

Thus, $\dfrac{7x+4}{(x+1)^2(3x+2)}=\dfrac{2}{x+1}+\dfrac{3}{(x+1)^2}-\dfrac{6}{3x+2}$

6. $\dfrac{1}{(x-1)^2(x+1)}$

Solution: Let $\dfrac{1}{(x-1)^2(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+1}$ ..... (i)

Multiplying by $(x-1)^2(x+1)$, we get

$1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$ ..... *(ii)*

$1 = A(x^2-1) + B(x+1) + C(x^2-2x+1)$ ..... *(iii)*

Put $x - 1 = 0$ i.e; $x = 1$ in *(ii)*

$1 = B(1+1)$

$1 = 2B$

$\Rightarrow B = \dfrac{1}{2}$

Put $x + 1 = 0$ i.e; $x = -1$ in *(ii)*

$1 = C(-1-1)^2$

$1 = C(-2)^2$

$1 = 4C$

$$C = \frac{1}{4}$$

Comparing co-efficients of x^2 in (iii)

$0 = A + C$

Put $C = \frac{1}{4}$ in it

$$0 = A + \frac{1}{4}$$

$$A = -\frac{1}{4}$$

Putting values of A, B, C in (i)

Partial fractions are

$$\frac{-1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{1}{4(x+1)}$$

$$= \frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2}$$

Thus, $\dfrac{1}{(x-1)^2(x+1)} = \dfrac{1}{4(x+1)} - \dfrac{1}{4(x-1)} + \dfrac{1}{2(x-1)}$

7. $\dfrac{3x^2 + 15x + 16}{(x+2)^2}$

Solution: $\dfrac{3x^2 + 15x + 16}{(x+2)^2}$

$$= \frac{3x^2 + 15x + 16}{x^2 + 4x + 4}$$

$$\begin{array}{r|l} & \quad 3 \\ \hline x^2 + 4x + 4 & 3x^2 + 15x + 16 \\ & \underline{-3x^2 \pm 12x \pm 12} \end{array}$$

$$= 3 + \frac{3x+4}{x^2+4x+4}$$

$3x + 4$

$$= 3 + \frac{3x+4}{(x+2)^2} \text{..........(i)}$$

Now, we take up $\frac{3x+4}{x^2+4x+4}$

$$\frac{3x+4}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} \quad \text{...... (ii)}$$

Multiplying by $(x+2)^2$, we get

$3x + 4 = A(x+2) + B$ ...... (iii)

Put $x + 2 = 0$ i.e; $x = -2$ in (iii)

$3(-2) + 4 = A(-2+2) + B$

$-6 + 4 = B$

$B = -2$

Comparing co-efficients of x in (iii)

$3 = A$

Putting values of A, B in (ii) joining with (i)

Partial fractions are

$$3 + \frac{3}{x+2} - \frac{2}{(x+2)^2}$$

Thus, $3 + \frac{3x+4}{(x+2)^2} = 3 + \frac{3}{x+2} - \frac{2}{(x+2)^2}$

8. $\frac{1}{(x^2-1)(x+1)}$

Solution: $\frac{1}{(x^2-1)(x+1)}$

$$= \frac{1}{(x-1)(x+1)(x+1)}$$

$$= \frac{1}{(x-1)(x+1)^2} \qquad \text{Or} \qquad = \frac{1}{(x+1)^2(x-1)}$$

Let $\dfrac{1}{(x+1)^2(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-1}$ .... (i)

Multiplying by $(x+1)^2(x-1)$, we get

$1 = A(x+1)(x-1) + B(x-1) + C(x+1)^2$ .... (ii)

$1 = A(x^2-1) + B(x-1) + C(x^2+2x+1)$ .... (iii)

Put $x+1=0$ i.e; $x=-1$ in (ii)

$1 = B(-1-1)$

$1 = -2B$

$$B = -\frac{1}{2}$$

Put $x-1=0$ i.e; $x=1$ in (ii)

$1 = C(1+1)^2$

$1 = C(2)^2$

$1 = 4C$

$$C = \frac{1}{4}$$

Comparing co-efficients of x^2 in (iii)

$0 = A + C$

Put $C = \dfrac{1}{4}$ in it

$$0 = A + \frac{1}{4}$$

$$A = -\frac{1}{4}$$

Putting values of A, B, C, in (i)

Partial fractions are

$$-\frac{1}{4(x+1)} - \frac{1}{2(x+1)^2} + \frac{1}{4(x-1)}$$

$$= \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x+1)^2}$$

Thus. $\frac{1}{(x+1)^2(x-1)} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x+1)^2}$

EXERCISE 4.3

Resolve into partial fractions.

1. $\frac{3x-11}{(x+3)(x^2+1)}$

Solution: Let $\frac{3x-11}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$ .... (i)

Multiplying by $(x+3)(x^2+1)$, we get

$3x - 11 = A(x^2 + 1) + (Bx + C)(x + 3)$ .... (ii)

$3x - 11 = A(x^2 + 1) + Bx^2 + 3Bx + Cx + 3C$

$3x - 11 = Ax^2 + A + Bx^2 + 3Bx + Cx + 3C$

$3x - 11 = Ax^2 + Bx^2 + 3Bx + Cx + A + 3C$

$3x - 11 = (A + B)\,x^2 + (3B + C)\,x + A + 3C$.....(iv)

Put $x + 3 = 0$ i.e; $x = -3$ in (ii)

$3(-3) - 11 = A\left[(-3)^2 + 1\right]$

$-9 - 11 = A[9 + 1]$

$-20 = 10A$

$A = -2$

Comparing co-efficients of x^2

$0 = A + B$

Put $A = -2$ in it

$0 = -2 + B$

$B = 2$

Comparing co-efficients of x in (iv)

$3 = 3B + C$

Put $B = 2$ in it

$3 = 3(2) + C$

$3 - 6 = C$

$\boxed{c = -3}$

Putting values of A, B, C in (i)

Partial fractions are

$-\dfrac{2}{x+3} + \dfrac{2x-6}{x^2+1}$

Thus. $\dfrac{3x-11}{(x+3)(x^2+1)} = -\dfrac{2}{x+3} + \dfrac{2x-3}{x^2+1}$

2. $\dfrac{3x+7}{(x^2+1)(x+3)}$

Solution: $\dfrac{3x+7}{(x^2+1)(x+3)}$

$$= \frac{3x+7}{(x+3)(x^2+1)}$$

Let $\dfrac{3x+7}{(x+3)(x^2+1)} = \dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+1}$ .... (i)

Multiplying by $(x+3)(x^2+1)$, we get

$3x+7 = A(x^2+1) + (Bx+C)(x+3)$ .... (ii)

$3x+7 = Ax^2 + A + Bx^2 + 3Bx + Cx + 3C$

$3x+7 = Ax^2 + Bx^2 + 3Bx + Cx + A + 3C$

$3x+7 = (A+B)x^2 + (3B+C)x + A + 3C$ .... (iii)

Put $x+3 = 0$ i.e; $x = -3$ in (ii)

$3(-3)+7 = A\{(-3)^2+1\}$

$-9+7 = A(9+1)$

$-2 = 10A$

$$A = -\frac{2}{10}$$

$$A = -\frac{1}{5}$$

Comparing coefficients of x^2 in (iii)

$A + B = 0$

Put $A = -\dfrac{1}{5}$ in it

$$-\frac{1}{5}+B=0$$

$$B=\frac{1}{5}$$

Comparing coefficients of x in (iii)

$3 = 3B + C$

Put $B=\frac{1}{5}$ in it

$$3=3\left(\frac{1}{5}\right)+C$$

$$3=\frac{3}{5}+C$$

$$3-\frac{3}{5}=C$$

$$C=\frac{15-3}{5}$$

$$C=\frac{12}{5}$$

Putting values of A, B, C in (i)

Partial fractions are

$$\frac{1}{5(x+3)}+\frac{\frac{1}{5}x+\frac{12}{5}}{x^2+1}$$

$$\frac{1}{5(x+3)}+\frac{x+12}{5(x^2+1)}$$

Thus. $\dfrac{3x+7}{(x+3)(x^2+1)} = -\dfrac{1}{5(x+3)} + \dfrac{x+12}{5(x^2+1)}$

3. $\dfrac{1}{(x+1)(x^2+1)}$

Solution: Let $\dfrac{1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+c}{x^2+1}$ .....**(i)**

Multiplying by $(x+1)(x^2+1)$, we get

$1 = A(x^2+1) + (Bx+C)(x+1)$ .... **(ii)**

$1 = Ax^2 + A + Bx^2 + Bx + Cx + C$

$1 = Ax^2 + Bx^2 + Bx + Cx + A + C$

$1 = (A+B)x^2 + (B+C)x + A + C$ .... **(iii)**

Put $x+1 = 0$ i.e; $x = -1$ in **(ii)**

$1 = A\left((-1)^2+1\right)$

$1 = A(1+1)$

$1 = 2A$

$A = \dfrac{1}{2}$

Comparing co-efficients of x^2 in (iii)

$0 = A + B$

Put $A = \dfrac{1}{2}$ in it

$0 = \dfrac{1}{2} + B$

$B = -\dfrac{1}{2}$

Comparing coefficients of x in (iii)

$0 = B + C$

Put $B = -\frac{1}{2}$ in it

$0 = -\frac{1}{2} + c$

$C = \frac{1}{2}$

C

Putting values of A, B, C in (i)

Partial fractions are

$$\frac{1}{2(x+1)} + \frac{-\frac{1}{2}x + \frac{1}{2}}{(x^2+1)}$$

$$= \frac{1}{2(x+1)} - \frac{x-1}{2(x^2+1)}$$

Thus, $$\frac{1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} - \frac{x-1}{2(x^2+1)}$$

4. $$\frac{9x-7}{(x+3)(x^2+1)}$$

Solution: Let $\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$ ...... (i)

Multiplying by $(x+3)(x^2+1)$, we get

$9x - 7 = A(x^2+1) + (Bx+C)(x+3)$ ...... (ii)

$9x - 7 = Ax^2 + A + Bx^2 + 3Bx + Cx + 3C$

$9x - 7 = (A+B)x^2 + (3B+C)x + (A+3C)$ ..... (iii)

Put $x + 3 = 0$ i.e: $x = -3$ in (ii)

$9(-3) - 7 = A((-3)^2 + 1)$

$$-27-7 \quad A(9+1)$$

$$-34 \quad 10A$$

$$A = \frac{-34}{10}$$

$$A = -\frac{17}{5}$$

Comparing coefficients of x^2 in (iii)

$$0 \quad A + B$$

Put $A = -\frac{17}{5}$ in it

$$0 = -\frac{17}{5} + B$$

$$B = \frac{17}{5}$$

Comparing coefficients of x in (iii)

$$9 \quad 3B + C$$

Putting $B = \frac{17}{5}$ in it.

$$9 \quad 3\left(\frac{17}{5}\right) + C$$

$$9 \quad \frac{51}{5} + C$$

$$C \quad 9 - \frac{51}{5}$$

$$= \frac{45 - 51}{5}$$

$C = -\dfrac{6}{5}$

Putting values of A, B, C in (i)

Partial fractions are

$$-\frac{17}{5(x+3)}+\frac{\frac{17}{5}x-\frac{6}{5}}{x^2+1}$$

$$=-\frac{17}{5(x+3)}+\frac{17x-6}{5(x^2+1)}$$

Thus, $\dfrac{9x-7}{(x+3)(x^2+1)}=\dfrac{17}{5(x+3)}+\dfrac{17x-6}{5(x^2+1)}$

5. $\dfrac{3x+7}{(x+3)(x^2+4)}$

Solution: Let $\dfrac{3x+7}{(x+3)(x^2+4)}=\dfrac{A}{x+3}+\dfrac{Bx+c}{x^2+4}$ .... (i)

Multiplying by $(x + 3)(x^2 + 4)$, we get

$3x + 7 = A(x^2 + 4) + (Bx + C)(x + 3)$ .... (ii)

$3x + 7 = Ax^2 + 4A + Bx^2 + 3Bx + Cx + 3C$

$3x + 7 = Ax^2 + Bx^2 + 3Bx + Cx + 4A + 3C$

$3x + 7 = (A + B)x^2 + (3B + C)x + 4A + 3C$ ...... (iii)

Put $x = -3$ in (ii)

$3(-3) + 7 = A\left[(-3)^2 + 4\right]$

$-9 + 7 = A(9 + 4)$

$-2 = 13A$

$A = -\dfrac{2}{13}$

Comparing coefficients of x^2 in (iii)

$0 = A + B$

Put $A = -\frac{2}{13}$ in it.

$$0 = -\frac{2}{13} + B$$

$$B = \frac{2}{13}$$

Comparing coefficients of x in (iii)

$3 = 3B + C$

Put $B = \frac{2}{13}$ in it.

$$3 = 3\left(\frac{2}{13}\right) + C$$

$$C = 3 - \frac{6}{13}$$

$$C = \frac{39-6}{13}$$

$$C = \frac{33}{13}$$

Putting values of A, B, C in (i)

Partial fractions are

$$= -\frac{2}{13(x+3)} + \frac{\frac{2}{13}x + \frac{33}{13}}{(x^2+4)}$$

$$= -\frac{2}{13(x+3)} + \frac{2x+33}{13(x^2+4)}$$

Thus, $\dfrac{3x+7}{(x+3)(x^2+4)} = -\dfrac{2}{13(x+3)} + \dfrac{2x+33}{13(x^2+4)}$

6. $\dfrac{x^2}{(x+2)(x^2+4)}$

Solution: Let $\dfrac{x^2}{(x+2)(x^2+4)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+4}$ ...... **(i)**

Multiplying by $(x+2)(x^2+4)$, we get

$x^2 = A(x^2+4)+(Bx+C)(x+2)$ .... **(ii)**

$x^2 = Ax^2+4A+Bx^2+2Bx+Cx+2C$

$x^2 = (A+B)x^2+(2B+C)x+(4A+2C)$ .... **(iii)**

Put $x+2=0$ i.e; $x=-2$ in **(ii)**

$(-2)^2 = A\left[(-2)^2+4\right]$

$4 = A(4+4)$

$4 = 8A$

$A = \dfrac{4}{8}$

$A = \dfrac{1}{2}$

Comparing coefficients of x^2 in **(iii)**

$1 = A+B$

Put $A = \dfrac{1}{2}$ in it.

$1 = \dfrac{1}{2}+B$

$B = 1-\dfrac{1}{2} = \dfrac{2-1}{2} = \dfrac{1}{2}$

$B = \frac{1}{2}$

Comparing coefficients of x in (iii)

$0 = 2B + C$

Put $B = \frac{1}{2}$ in it.

$0 = 2\left(\frac{1}{2}\right) + C$

$0 = 1 + C$

$C = -1$

Putting values of A, B, C in (i)

$$\frac{1}{2(x+2)} + \frac{\frac{1}{2}x - 1}{x^2 + 4}$$

$$= \frac{1}{2(x+2)} + \frac{x-2}{2(x^2+4)}$$

Thus, $\frac{x^2}{(x+2)(x^2+4)} = \frac{1}{2(x+2)} + \frac{x-2}{2(x^2+4)}$

7. $\frac{1}{x^3+1}$ [**Hint** $\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$]

Solution: $\frac{1}{x^3+1}$

$$= \frac{1}{(x+1)(x^2-x+1)}$$

Let $\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ ...... (i)

Multiplying by $(x+1)(x^2-x+1)$, we get

$1 = A(x^2-x+1)+(Bx+C)(x+1)$ .... (ii)

$1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$

$1 = Ax^2 + Bx^2 + Bx + Cx - Ax + A + C = 0$

$1 = (A+B)x^2 + (B+C-A)x + A + C = 0$ ...... (iii)

Put $x+1=0$ i.e; $x=-1$ in (ii)

$1 = A\left[(-1)^2-(-1)+1\right]$

$1 = A(1+1+1)$

$1 = 3A$

$$A = \frac{1}{3}$$

Comparing coefficients of x^2 in (iii)

$0 = A + B$

Put $A = \frac{1}{3}$ in it.

$$0 = \frac{1}{3} + B$$

$$B = -\frac{1}{3}$$

Comparing constant terms in (iii)

$1 = A + C$

Put $A = \frac{1}{3}$ in it.

$$1 = \frac{1}{3} + C$$

$$C = 1 - \frac{1}{3}$$
$$= \frac{3-1}{3}$$
$$C = \frac{2}{3}$$

Putting values of A, B, C in (i)

Partial fractions are

$$\frac{1}{3(x+1)} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2 - x + 1}$$
$$= \frac{1}{3(x+1)} - \frac{x-2}{3(x^2 - x + 1)}$$

Thus. $$\frac{1}{(x+1)(x^2 - x + 1)} = \frac{1}{3(x+1)} - \frac{x-2}{3(x^2 - x + 1)}$$

8. $\dfrac{x^2+1}{x^3+1}$

Solution: $$\frac{x^2+1}{x^3+1}$$
$$= \frac{x^2+1}{(x+1)(x^2 - x + 1)}$$

Let $$\frac{x^2+1}{(x+1)(x^2 - x + 1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2 - x + 1} \quad \text{......(i)}$$

Multiplying by $(x+1)(x^2 - x + 1)$, we get

$$x^2 + 1 = A(x^2 - x + 1) + (Bx + C)(x + 1) \quad \text{.......(ii)}$$
$$x^2 + 1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$x^2 + 1 = Ax^2 + Bx^2 + Bx + Cx - Ax + A + C$$

$$x^2 + 1 = (A+B)x^2 + (B+C-A)x + A + C$$

Put $x + 1 = 0$ i.e; $x = -1$ in (ii)

$$(-1)^2 + 1 = A\left[(-1)^2 - (-1) + 1\right]$$

$$1 + 1 = A(1+1+1)$$

$$2 = 3A$$

$$A = \frac{2}{3}$$

Comparing coefficients of x^2 in (iii)

$$1 = A + B$$

Put $A = \frac{2}{3}$ in it.

$$1 = \frac{2}{3} + B$$

$$B = 1 - \frac{2}{3}$$

$$B = \frac{3-2}{3}$$

$$B = \frac{1}{3}$$

Comparing constant terms in (iii)

$$1 = A + C$$

Put $A = \frac{2}{3}$ in it.

$$1 = \frac{2}{3} + C$$

$$C = 1 - \frac{2}{3}$$

$$C = \frac{3-2}{3}$$

$$C = \frac{1}{3}$$

Putting $A = \frac{2}{3}, B = \frac{1}{3}, C = \frac{1}{3}$ in (i)

Partial fractions are

$$\frac{2}{3(x+1)} + \frac{\frac{1}{3}x + \frac{1}{3}}{(x^2 - x + 1)}$$

$$= \frac{2}{3(x+1)} + \frac{x+1}{3(x^2 - x + 1)}$$

Thus. $\dfrac{x^2+1}{(x+1)(x^2-x+1)} = \dfrac{2}{3(x+1)} + \dfrac{x+1}{3(x^2-x+1)}$

EXERCISE 4.4

Resolve into partial fractions.

1. $\dfrac{x^3}{(x^2+4)^2}$

Solution: Let $\dfrac{x^3}{(x^2+4)^2} = \dfrac{Ax+B}{x^2+4} + \dfrac{Cx+D}{(x^2+4)^2}$ ..... (i)

Multiplying by $(x^2+4)^2$, we get

$$x^3 = (Ax+B)(x^2+4) + (Cx+D)$$

$$x^3 = Ax^3 + 4Ax + Bx^2 + 4B + Cx + D$$

$x^3 = Ax^3 + Bx^2 + (4A + C)x + (4B + D)$

Comparing co-efficients of x^3 in……(ii)

1 A

$A = 1$

Comparing coefficients of x^2 in (ii)

0 B

$B = 0$

Comparing co-efficient x in (ii)

0 4A + C

Put A 1 in it

0 4(1) + C

$C = -4$

Comparing constant terms in (ii)

0 4B + D

Put B 0 in it

0 0 + D

$D = 0$

Putting values of A, B, C, D in (i)

$$\frac{x+0}{x^2+4} + \frac{-4x+0}{(x^2+4)^2}$$

$\frac{x}{x^2+4} - \frac{4x}{(x^2+4)^2}$ are partial fractions.

Hence, $$\frac{x^3}{(x^2+4)^2} = \frac{x}{x^2+4} - \frac{4x}{(x^2+4)^2}$$

2. $\dfrac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}$

Solution: Let $\dfrac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \dfrac{A}{x+1} + \dfrac{Bx+C}{(x^2+1)} + \dfrac{Dx+E}{(x^2+1)^2} \ldots(M)$

Multiplying by $(x+1)(x^2+1)^2$, we get

$$x^4+3x^2+x+1 = A(x^2+1)^2 + (Bx+C)(x+1)(x^2+1) + (Dx+E)(x+1)\ldots(P)$$

$$x^4+3x^2+x+1 = A(x^4+2x^2+1) + (Bx+C)(x^3+x+x^2+1) + Dx^2+Dx+Ex+E$$

$$x^4+3x^2+x+1 = Ax^4+2Ax^2+A+Bx^4+Bx^2+Bx^3 + Bx+Cx^3+Cx+Cx^2+C+Dx^2+Dx+Ex+E$$

$$x^4+3x^2+x+1 = (A+B)x^4+(B+C)x^3+(2A+B+C+D)x^2+(B+C+D+E)x+A+C+E$$

Put $x = -1$ in (P)

$(-1)^4 + 3(-1)^2 + (-1) + 1 = A((-1)^2+1)^2$

$1 + 3(1) - 1 + 1 = A(2)^2$

$4 = 4A$

$\therefore A = 1$

Comparing co-efficients of x^4, x^3, x^2, x and constant terms

$A + B = 1 \ldots\ldots\ldots\ldots(i)$

$B + C = 0 \ldots\ldots\ldots\ldots(ii)$

$2A + B + C + D = 3 \ldots\ldots(iii)$

$A + C + E = 1 \ldots\ldots\ldots\ldots(iv)$

Put $A = 1$ in (i)

$1 + B = 1$

$B = 0$

Put B = 0 in (ii)

$0 + C = 0$

$\boxed{C = 0}$

Put values of A, B, C in (iii)

$2(1) + 0 + 0 + D = 3$

$2 + D = 3$

$D = 3 - 2$

$D = 1$

Put A, C values in (iv)

$1 + 0 + E = 1$

$1 + E = 1$

$E = 1 - 1$

$\boxed{E = 0}$

Putting values of A, B, C, D, E in (M)

Partial fractions are

$$\frac{1}{x+1}+\frac{0+0}{x^2+1}+\frac{x+0}{(x^2+1)^2}$$

$$=\frac{1}{x+1}+\frac{x}{(x^2+1)^2}$$

Hence, $\dfrac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}=\dfrac{1}{x+1}+\dfrac{x}{(x^2+1)^2}$

3. $\dfrac{x^2}{(x+1)(x^2+1)^2}$

Solution: Let $\dfrac{x^2}{(x+1)(x^2+1)^2}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}\;...(i)$

Multiplying by $(x+1)(x^2+1)^2$, we get

$$x^2 = A(x^2+1)^2 + (Bx+C)(x+1)(x^2+1)$$
$$+(Dx+E)(x+1)\ldots\ldots(ii)$$
$$x^2 = A(x^4+2x^2+1)+(Bx+C)(x^3+x+x^2+1)$$
$$+Dx^2+Dx+Ex+E$$
$$x^2 = Ax^4+2Ax^2+A+Bx^4+Bx^2+Bx^3+Bx$$
$$+Cx^3+Cx+Cx^2+C+Dx^2+Dx+Ex+E$$
$$x^2 = (A+B)x^4+(B+C)x^3+(2A+B+C+D)x^2$$
$$+(B+C+D+E)x+A+C+E$$

Put $x = -1$ in (ii)

$(-1)^2 = A((-1)^2+1)^2$

$1 = A(1+1)^2$

$1 = A(2)^2$

$1 = 4A$

$A = \dfrac{1}{4}$

Comparing the co-efficients of x^4, x^3, x^2, x and constant terms.

$A + B = 0\ldots\ldots\ldots\ldots\ldots\ldots(iii)$

$B + C = 0\ldots\ldots\ldots\ldots\ldots\ldots(iv)$

$2A + B + C + D = 1\ldots\ldots\ldots(v)$

$B + C + D + E = 0\ldots\ldots\ldots(vi)$

$A + C + E = 0\ldots\ldots(vii)$

Put $A = \dfrac{1}{4}$ in (iii)

$\dfrac{1}{4} + B = 0$

$$B = -\frac{1}{4}$$

Put $B = -\frac{1}{4}$ in (iv)

$$-\frac{1}{4} + C = 0$$

$$C = \frac{1}{4}$$

Put values of A, C in (vii)

$$\frac{1}{4} + \frac{1}{4} + E = 0$$

$$E = -\frac{1}{4} - \frac{1}{4}$$

$$E = -\frac{1}{2}$$

Put values of B, C in (vi)

$$D + E = 0$$

$$D = -E$$

Put $E = -\frac{1}{2}$ in it

$$D = -\left(-\frac{1}{2}\right)$$

$$D = +\frac{1}{2}$$

Putting values of A, B, C, D, E in (i)

$$\frac{1}{4(x+1)}+\frac{-\frac{1}{4}x+\frac{1}{4}}{(x^2+1)}+\frac{\frac{1}{2}x-\frac{1}{2}}{(x^2+1)^2}$$

$$\frac{1}{4(x+1)}-\frac{x-1}{4(x^2+1)}+\frac{x-1}{2(x^2+1)^2} \quad \text{(Partial Fractions)}$$

Hence, $\frac{x^2}{(x+1)(x^2+1)^2}=\frac{1}{4(x+1)}-\frac{x-1}{4(x^2+1)}+\frac{x-1}{2(x^2+1)^2}$

4. $\frac{x^2}{(x-1)(x^2+1)^2}$

Solution: Let $\frac{x^2}{(x-1)(x^2+1)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2} \ldots \text{(i)}$

Multiplying by $(x-1)(x^2+1)^2$, we get

$x^2=A(x^2+1)^2+(Bx+C)(x-1)(x^2+1)+(Dx+E)(x-1) \ldots \text{(ii)}$

$x^2=A(x^4+2x^2+1)+(Bx+C)(x^3-x-x^2-1)$
$+Dx^2-Dx+Ex-E$

$x^2=Ax^4+2Ax^2+A+Bx^4-Bx^2-Bx^3-Bx$
$+Cx^3-Cx-Cx^2-C+Dx^2-Dx+Ex-E=0$

$x^2=(A+B)x^4-(B-C)x^3+(2A-B-C+D)x^2$
$-(B+C+D-E)x+(A-C-E)$

Comparing coefficients of x^4, x^3, x^2, x and constant terms.

$0=(A+B) \ldots\ldots\ldots\ldots \text{(iii)}$

$0=-B+C \ldots\ldots\ldots\ldots \text{(iv)}$

$1=2A-B-C+D \ldots\ldots \text{(v)}$

$0=-B-C-D+E \ldots\ldots \text{(vi)}$

$0=A-C-E \ldots\ldots\ldots\ldots \text{(vii)}$

Put $x-1=0$ i.e; $x=1$ in (ii)

$(1)^2 = A((1)^2 + 1)^2$

$1 = A(1 + 1)^2$

$1 = A(2)^2$

$1 = 4A$

$$A = \frac{1}{4}$$

$$\frac{1}{4} + B = 0 \quad \text{from (iii)}$$

$$B = -\frac{1}{4}$$

$$-\left(-\frac{1}{4}\right) + C = 0 \quad \text{from (iv)}$$

$$\frac{1}{4} + C = 0$$

$$C = -\frac{1}{4}$$

Put values of A, B, C in (vii)

$$\frac{1}{4} + \frac{1}{4} - E = 0$$

$$\frac{2}{4} - E = 0$$

$$E = \frac{2}{4} = \frac{1}{2}$$

$$0 = \frac{1}{4} + \frac{1}{4} - D + \frac{1}{2} \qquad \text{from (vi)}$$

$$0 = \frac{1}{2} - D + \frac{1}{2}$$

$$D = \frac{1}{2} + \frac{1}{2} = 1$$

Putting values of A, B, C, D, E in (i)

$$\frac{1}{4(x-1)} + \frac{-\frac{1}{4}x - \frac{1}{4}}{x^2+1} + \frac{x + \frac{1}{2}}{(x^2+1)^2}$$

$$= \frac{1}{4(x-1)} - \frac{x+1}{4x^2+1} + \frac{2x+1}{2(x^2+1)^2} \quad \textbf{(Partial Fractions)}$$

Hence, $\dfrac{x^2}{(x-1)(x^2+1)^2} = \dfrac{1}{4(x-1)} - \dfrac{x+1}{4(x^2+1)} + \dfrac{2x+1}{2(x^2+1)^2}$

5. $\dfrac{x^4}{(x^2+2)^2}$

Solution: $\dfrac{x^4}{(x^2+2)^2}$

$$= \frac{x^4}{x^4+4x^2+4}$$

$$\begin{array}{r|l} & \quad 1 \\ \hline x^4+4x^2+4 & x^4 \\ & \underline{-x^4 \pm 4x^2 \pm 4} \\ & \quad -4x^2-4 \end{array}$$

$$= 1 - \frac{4x^2+4}{x^4+4x^2+4}$$

$$= 1 - \frac{4x^2+4}{(x^2+2)^2}$$

Now, we take up $\dfrac{4x^2+4}{(x^2+2)^2}$

Let $\dfrac{4x^2+4}{(x^2+2)^2} = \dfrac{Ax+B}{(x^2+2)} + \dfrac{Cx+D}{(x^2+2)^2}$ .... (i)

Multiplying by $(x^2+2)^2$, we get

$4x^2+4=(Ax+B)(x^2+2)+Cx+D$

$4x^2+4=Ax^3+2Ax+Bx^2+2B+Cx+D.$

$4x^2+4=Ax^3+Bx^2+(2A+C)x+2B+D$

Comparing coefficients of x^3, x^2, x and constant terms.

$A = 0$ ....(ii) (Co-eff of x^3)

$B = 4$ ....(iii) (Co-eff of x^2)

$0=2A+C$.....(iv) (Co-eff of x)

$4=2B+D$......(v) (Constants)

Put $A=0$ in (iv)

$0=0+C$

$C = 0$

Put $B=4$ in (v)

$4=2(4)+D$

$4=8+D$

$D=4-8$

$\boxed{D=-4}$

Putting values of A, B, C, D in (i)

$$\frac{4x^2+4}{(x^2+2)^2}=\frac{0+4}{x^2+2}+\frac{0-4}{(x^2+2)^2}$$

$$\frac{4x^2+4}{(x^2+2)^2}=\frac{4}{x^2+2}-\frac{4}{(x^2+2)^2}$$

Thus, $1-\dfrac{4x^2+4}{(x^2+2)^2}=1-\dfrac{4}{x^2+2}+\dfrac{4}{(x^2+2)^2}$ (Partial Fractions)

6. $\dfrac{x^5}{(x^2+1)^2}$

Solution: $\dfrac{x^5}{(x^2+1)^2}$

$$= \frac{x^5}{x^4+2x^2+1}$$

$$\begin{array}{r|l} & \quad x \\ \hline x^4+2x^2+1 & x^5 \\ & \pm x^5 \pm 2x^3 \pm x \\ \hline & \quad -2x^3 - x \end{array}$$

$$= x - \frac{2x^3+x}{x^4+2x^2+1}$$

$$= x - \frac{2x^3+x}{(x^2+1)^2}$$

Now we take up

$$= \frac{2x^3+x}{(x^2+1)^2}$$

Let $\dfrac{2x^3+x}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$ ....(M)

Multiplying by $(x^2+1)^2$, we get

$$2x^3+x = (Ax+B)(x^2+1)+Cx+D$$

$$2x^3+x = Ax^3+Ax+Bx^2+B+Cx+D$$

$$2x^3+x = Ax^3+Bx^2+(A+C)x+B+D$$

Comparing co-efficients of x^3, x^2, x and constant terms.

$A = 2$ ....(i)

$B = 0$ ....(ii)

A + C = 1........(iii)

B + D = 0........(iv)

Put A = 2 in (iii)

$2 + C = 1$

$C = 1 - 2$

$C = -1$

Put B = 0 in (iv)

$0 + D = 0$

$D = 0$

Putting values of A, B, C, D in (M)

$$\frac{2x^3 + x}{(x^2+1)^2} = \frac{2x+0}{x^2+1} + \frac{-1x+0}{(x^2+1)^2}$$

$$\frac{2x^3 + x}{(x^2+1)^2} = \frac{2x}{x^2+1} - \frac{x}{(x^2+1)^2}$$

Thus, $$x - \frac{2x^3 + x}{(x^2+1)^2} = x - \left(\frac{2x}{x^2+1} - \frac{x}{(x^2+1)^2}\right)$$

$$= x - \frac{2x}{x^2+1} + \frac{x}{(x^2+1)^2} \text{ (Partial Fractions)}$$

EXERCISE 5.1

**1. If $X = \{1, 4, 7, 9\}$ and $Y = \{2, 4, 5, 9\}$
Then find:**

(i) $X \cup Y$

Solution: $X \cup Y = \{1, 4, 7, 9\} \cup \{2, 4, 5, 9\}$
$= \{1, 2, 4, 5, 7, 9\}$

(ii) $X \cap Y$

Solution: $X \cap Y = \{1, 4, 7, 9\} \cap \{2, 4, 5, 9\}$
$= \{4, 9\}$

(iii) $Y \cup X$

Solution: $Y \cup X = \{2, 4, 5, 9\} \cup \{1, 4, 7, 9\}$
$= \{1, 2, 4, 5, 7, 9\}$

(iv) $Y \cap X$

Solution: $Y \cap X = \{2, 4, 5, 9\} \cap \{1, 4, 7, 9\}$
$= \{4, 9\}$

2. If X = Set of prime numbers less than or equal to 17 and Y= Set of first 12 natural numbers, then find the following

Here, $X = \{2, 3, 5, 7, 11, 13, 17\}$
$Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

(i) $X \cup Y$

$= \{2, 3, 5, 7, 11, 13, 17\} \cup \{1, 2, 3, \ldots 12\}$
$= \{1, 2, 3, \ldots\ldots 12, 13, 17\}$

(ii) $Y \cup X$

$= \{1,2,3,\ldots.12,13,17\} \cup \{2,3,5,7,11,13,17\}$
$= \{1,2,3,\ldots\ldots 12,13,17\}$

(iii) $X \cap Y$

$= \{2,3,5,7,11,13,17\} \cap \{1,2,3,\ldots\ldots,12\}$
$= \{2, 3, 5, 7, 11\}$

(iv) $Y \cap X$

$= \{1,2,3,\ldots,12\} \cap \{2,3,5,7,11,13,17\}$
$= \{2,3,5,7,11\}$

3. If $X = \phi$, $Y = Z^+$, $T = O^+$, then find

Solution: $X = \{\ \}$
$Y = \{0, 1, 2, 3, \ldots\ldots\}$
$T = \{1, 3, 5, \ldots\ldots\}$

(i) $X \cup Y$

$= \{\ \} \cup \{0, 1, 2, 3\ldots\ldots\}$
$= \{0, 1, 2, 3\ldots\ldots\} = Y$

(ii) $X \cup T$

$= \{\ \} \cup \{1, 3, 5\ldots\}$
$= \{1, 3, 5, \ldots\} = T$

(iii) $Y \cup T$

$= \{0,1,2,3,\ldots\} \cup \{1,3,5\ldots\ldots\}$
$= \{0,1,2,3\ldots\} = Y$

(iv) $X \cap Y$

$= \{\ \} \cap \{0, 1, 2, 3\ldots\ldots\}$
$= \{\ \}$

(v) $X \cap T$

$= \{\ \} \cap \{1, 3, 5,\ldots\ldots\}$
$= \{\ \}$

(vi) $Y \cap T$

$= \{0,1,2,3\ldots\ldots\} \cap \{1, 3, 5\ldots\ldots\}$
$= \{1,3,5\ldots\} = T$

4. If $U = \{x | x \in N \wedge 3 < x \leq 25\}$,
$X = \{x | x \text{ is prime} \wedge 8 < x < 25\}$
and $Y = \{x | x \in W \wedge 4 \leq x \leq 17\}$,
Find the value of:

(i) $(X \cup Y)'$

Solution: Here,

$U = \{4, 5, 6, 7, \ldots 25\}$
$X = \{11, 13, 17, 19, 23\}$
$Y = \{4, 5, 6, 7, \ldots 17\}$

Find $(X \cup Y)'$

Now $X \cup Y$

$= \{11,13,17,19,23\} \cup \{4,5,6,7, \ldots 17\}$
$= \{4,5,6,7, \ldots 17,19,23\}$

$(X \cup Y)'$

$= U - (X \cup Y)$
$= \{4,5, \ldots 25\} - \{4,5,6, \ldots 17,19,23\}$
$= \{18, 20, 21, 22, 24, 25\}$

(ii) $X' \cap Y'$

Now $X' = U - X$

$= \{4, 5, 6, \ldots 25\} - \{11,13,17,19,23\}$
$= \{4,5,6, \ldots ,10,12,14,15,16,18,20,21,22,24,25\}$

and $Y' = U - Y$

$= \{4,5,6, \ldots ,25\} - \{4,5,6,7, \ldots 17\}$
$= \{18,19,20,21,22,23,24,25\}$

Now $X' \cap Y'$

$= \{4,5, \ldots 10,12,14,15,16,18,20,21,22,24,25\}$
$\cap \{18,19,20, \ldots 25\}$
$= \{18, 20, 21, 22, 24, 25\}$

(iii) $(X \cup Y)'$

Now, $X \cap Y$

$= \{11,13,17,19,23\} \cap \{4, 5, 6, \ldots, 17\}$
$= \{11, 13, 17\}$

$(X \cap Y)'$

$= U - (X \cap Y)$
$= \{4,5,6, \ldots 25\} - \{11,13,17\}$

$= \{4,5,6\ldots10,12,14,15,16,18,19,20,21,22,2324,25\}$

(iv) $X' \cup Y'$

$X' = U - X$

$= \{4,5,6\ldots25\} - \{11,13,17,19,23\}$

$\{4,5,6\ldots10,12,14,15,16,18,19,20,21,22,23,24,25\}$

$Y' = U - Y$

$= \{4,5,6\ldots25\} - \{4,5,6\ldots17\}$

$= \{18,19,20,\ldots25\}$

Now $X' \cap Y'$

$= \{4,5,6\ldots10,12,14,15,16,18,20,21,22,24,25\}$

$\cup \{18, 19, 20, \ldots25\}$

$= \{4,5,6\ldots10,12,14,15,16,18,\ldots,25\}$

5. If $X = \{2, 4, 6, \ldots, 20\}$ and $Y = \{4, 8, 12, \ldots, 24\}$, then find the following:

(i) $X - Y$

Solution: $X - Y$

$= \{2, 4, 6,\ldots, 20\} - \{4, 8, 12,\ldots24\}$

$= \{2, 6, 10, 14, 18\}$

(ii) $Y - X$

$= \{4, 8, 12, \ldots, 24\} - \{2, 4, 6, \ldots, 20\}$

$= \{24\}$

6. If $A = N$ and $B = W$, then find the value of

(i) $A - B$

Solution: $= \{1, 2, 3,\ldots\} - \{0, 1, 2, 3\ldots\}$

$= \{\ \}$

(ii) $B - A$

$= \{0, 1, 2, 3\ldots\} - \{1, 2, 3\ldots\}$

$= \{0\}$

Properties of Union and Intersection

(i) Commutative property of union.

$$A \cup B = B \cup A$$

(ii) Commutative property of Intersection.

$$A \cap B = B \cap A$$

(iii) Associative property of union.

$$(A \cup B) \cup C = A \cup (B \cup C)$$

(iv) Associative property of intersection.

$$(A \cap B) \cap C = A \cap (B \cap C)$$

(v) Distributive property of union over intersection.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(vi) Distributive property of union over intersection

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(vii) De-Morgan's Laws.

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

EXERCISE 5.2

1. If $X = \{1, 3, 5, 7, ..., 19\}$,
$Y = \{0, 2, 4, 6, 8, ..., 20\}$
$Z = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$,
then find the following:

(i) $X \cup (Y \cup Z)$

Solution: $X \cup (Y \cup Z)$

$= \{1,3,5,7,...,19\} \cup [\{0,2,4,6,8,...20\} \cup \{2,3,5,7,11,13,17,19,23\}]$

$= \{1,3,5,7,...,19\} \cup \{0,2,3,4,...,8,10,11,12,13,14,16,17,18,19,20,23\}$

$= \{0,1,2,3,......,20,23\}$

(ii) $(X \cup Y) \cup Z$

Solution: $(X \cup Y) \cup Z$

$= [\{1,3,5,7,\ldots19\} \cup \{0,2,4,6,8\ldots20\}] \cup \{2,3,5,7,11,13,17,19,23\}$

$= \{0,1,2,3,\ldots.20\} \cup \{2,3,5,7,11,13,17,19,23\}$

$= \{0,1,2,3,\ldots.,20,23\}$

(iii) $X \cap (Y \cap Z)$

Solution: $X \cap (Y \cap Z)$

$= \{1,3,5,7\ldots,19\} \cap [\{0,2,4,6,\ldots20\} \cap \{2,3,5,7,11,13,17,19,23\}]$

$= \{1,3,5,7,\ldots19\} \cap \{2\}$

$= \{\ \ \}$

(iv) $(X \cap Y) \cap Z$

Solution: $(X \cap Y) \cap Z$

$= [\{1,3,5,7,\ldots,19\} \cap \{0,2,4,6\ldots,20\}] \cap \{2,3,5,7,11,13,17,19,23\}$

$= \{\ \ \} \cap \{2,3,5,7,11,13,15,17,19,23\}$

$= \{\ \ \}$

(v) $X \cup (Y \cap Z)$

Solution: $X \cup (Y \cap Z)$

$= \{1,3,5,7\ldots19\} \cup [\{0,2,4,6,8,\ldots20\} \cap \{2,3,5,7,11,13,15,17,19,23\}]$

$= \{1,3,5,7,\ldots,19\} \cup \{2\}$

$= \{1, 2, 3, 5, 7, \ldots.., 19\}$

(vi) $(X \cup Y) \cap (X \cup Z)$

Solution: $(X \cup Y) \cap (X \cup Z)$

$X \cup Y = \{1,3,5,7,\ldots,19\} \cup \{0,2,4,6,8,\ldots,20\}$

$= \{0,1,2,3,4,\ldots.20\}$ (i)

$X \cup Z = \{1,3,5,7,\ldots,19\} \cup \{2,3,5,7,11,13,17,19,23\}$

$= \{1,2,3,5,7,11,13,17,19,23\}$ (ii)

From (i), (ii)

$(X \cup Y) \cap (X \cup Z) = \{0,1,2,3,4,\ldots,20\} \cap \{1,2,3,5,7,11,13,17,19,23\}$

$= \{1,2,3,5,7,\ldots\ldots19\}$

(vii) $X \cap (Y \cup Z)$

Now, $Y \cup Z = \{0,2,4,6,8,\ldots,20\} \cup \{2,3,5,7,11,13,17,19,23\}$
$= \{0,2,3,4,\ldots,8,10,11,12,13,14,16,17,18,19,20,23\}$
$X \cap (Y \cup Z) = \{1,3,5,7,\ldots,19\} \cap \{0,2,3,4,\ldots,8,10,11,12,13,14,16,17,18,19,20,23\}$
$= \{3, 5, 7, 11, 13, 17, 19\}$

(viii) $(X \cap Y) \cup (X \cap Z)$

Solution: $X \cap Y$
$= \{1,3,5,7,\ldots,19\} \cap \{0,2,4,6,8,\ldots,20\}$
$= \{\ \}$ (i)
$X \cap Z$
$= \{1,3,5,7,11,\ldots,19\} \cap \{2,3,5,7,11,13,17,19,23\}$
$= \{3,5,7,11,13,17,19\}$ (ii)
Now, $(X \cap Y) \cup (X \cap Z)$
$= \{\ \} \cup \{3,5,7,11,13,17,19\}$
$= \{3, 5, 7, 11, 13, 17, 19\}$

2. If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8\}$, $C = \{1, 4, 8\}$.
Prove the following identities:

(i) $A \cap B = B \cap A$
L.H.S.
$A \cap B$
$= \{1,2,3,4,5,6\} \cap \{2,4,6,8\}$
$= \{2,4,6\}$ (i)
R.H.S
$B \cap A$
$= \{2,4,6,8\} \cap \{1,2,3,\ldots,6\}$
$= \{2,4,6\}$ (ii)
From (i), (ii)
$A \cap B = B \cap A$

(ii) $A \cup B = B \cup A$

L.H.S.

$A \cup B$

$= \{1,2,3,\ldots,6\} \cup \{2,4,6,8\}$

$= \{1,2,3,\ldots,6,8\}$ (i)

R.H.S.

$B \cup A$

$= \{2, 4, 6, 8\} \cup \{1, 2, 3,\ldots,6\}$

$= \{1, 2, 3, \ldots, 6, 8\}$ (ii)

From (i), (ii)

$A \cup B = B \cup A$

(iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

L.H.S.

$A \cap (B \cup C)$

$= \{1,2,3\ldots,6\} \cap [\{2,4,6,8\} \cup \{1,4,8\}]$

$= \{1,2,3,\ldots,6\} \cap \{1,2,4,6,8\}$

$= \{1,2,4,6\}$ (i)

R.H.S.

$(A \cap B) \cup (A \cap C)$

$= [\{1,2,3,\ldots,6\} \cap \{2,4,6,8\}] \cup [\{1,2,3,\ldots,6\} \cap \{1,4,8\}]$

$= \{2,4,6\} \cup \{1,4\}$

$= \{1,2,4,6\}$ (ii)

From (i), (ii)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iv) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

L.H.S.

$A \cup (B \cap C)$

$= \{1,2,3,\ldots,6\} \cup [\{2, 4, 6, 8\} \cap \{1, 4, 8\}]$

$= \{1, 2, 3,\ldots, 6\} \cup \{4, 8\}$

$= \{1, 2, 3,\ldots, 6, 8\}$ (i)

R.H.S.

$(A \cup B) \cap (A \cup C)$

$[\{1,2,3,\ldots 6\} \cup \{2, 4, 6, 8\}] \cap \{1,2,3\ldots,6\} \cup \{1,4,8\}$

$\{1, 2, 3, \ldots, 6, 8\} \cap \{1, 2, 3, \ldots, 6, 8\}$

$\{1, 2, 3, \ldots, 6, 8\}$

3. **If** $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 3, 5, 7, 9\}$, $B = \{2, 3, 5, 7\}$.

then verify the De-Morgan's Laws

i.e., $(A \cap B)' = A' \cup B'$ **and** $(A \cup B)' = A' \cap B'$

(i) $(A \cap B)' = A' \cup B'$

Solution: $(A \cap B)' \quad A' \cup B'$

L.H.S.

$A \cap B$

$\{1, 3, 5, 7, 9\} \cap \{2, 3, 5, 7\}$

$\{3, 5, 7\}$ (i)

$(A \cap B)'$

$U \quad (A \cap B)$

$\{1, 2, 3, \ldots, 10\} - \{3, 5, 7\}$ from (i)

$\{1, 2, 4, 6, 8, 9, 10\}$ (D)

R.H.S.

$A' \quad U \quad A$

$\{1, 2, 3, \ldots, 10\} \quad \{1, 3, 5, 7, 9\}$

$\{2, 4, 6, 8, 10\}$

$B' \quad U \quad B$

$\{1, 2, 3, \ldots, 10\} \quad \{2, 3, 5, 7\}$

$\{1, 4, 6, 8, 9, 10\}$

Now, $A' \cup B'$

$\{2, 4, 6, 8, 10\} \cup \{1, 4, 6, 8, 9, 10\}$

$\{1, 2, 4, 6, 8, 9, 10\}$ (E)

From (D), (E)

$(A \cap B)' = A' \cup B'$

(ii) $(A \cup B)' = A' \cap B'$

L.H.S.

$A \cup B$

$= \{1, 3, 5, 7, 9\} \cup \{2, 3, 5, 7\}$

$= \{1, 2, 3, 5, 7, 9\}$

$(A \cup B)'$

$= \{1, 2, 3, \ldots, 10\} - \{1, 2, 3, 5, 7, 9\}$

$= \{4, 6, 8, 10\}$ (F)

R.H.S.

$A' \cap B'$

$= \{2, 4, 6, 8, 10\} \cap \{1, 4, 6, 8, 9, 10\}$

$= \{4, 6, 8, 10\}$ (J)

From (F), (J)

$(A \cup B)' = A' \cap B'$

4. If $U = \{1, 2, 3, \ldots, 20\}$,
$X = \{1, 3, 7, 9, 15, 18, 20\}$
and $Y = \{1, 3, 5, \ldots, 17\}$, then show that

(i) $X - Y = X \cap Y'$

Solution: Let us find X', Y'

$X' = U - X$

$= \{1, 2, 3, \ldots, 20\} - \{1, 3, 7, 9, 15, 18, 20\}$

$= \{2, 4, 5, 6, 8, 10, 11, 12, 13, 14, 16, 17, 19\}$

$Y' = U - Y$

$= \{1, 2, 3, \ldots, 20\} - \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$

$= \{2, 4, 6, 8, 10, 12, 14, 16, 18, 19, 20\}$

Now, $X - Y$

$= \{1,3,7,9,15,18,20\} - \{1,3,5,7,9,11,13,15,17\}$

$= \{18, 20\}$ (R)

and $X \cap Y'$

$= \{1,3,7,9,15,18,20\} \cap \{2,4,6,8,10,12,14,16,18,19,20\}$
$= \{18, 20\}$ (Q)
From (Q), (R)
$X - Y = X \cap Y'$

(ii) $\boldsymbol{Y - X = Y \cap X'}$

Solution: To show $Y - X = Y - X'$
L.H.S.
$Y - X$
$= \{1,3,5,7,9,11,13,15,17\} - \{1,3,7,9,15,18,20\}$
$= \{5, 11, 13, 17\}$ (Q)
R.H.S.
$Y \cap X'$
$= \{1,3,5,7,9,11,13,15,17\} - \{2,4,5,6,8,10,11,12,13,14,16,17,19\}$
$= \{5, 11, 13, 17\}$ (R)
From (Q), (R)
$Y - X = Y \cap X'$

EXERCISE 5.3

1. **If** $\boldsymbol{U = \{1, 2, 3, 4, ..., 10\}}$,
$\boldsymbol{A = \{1, 3, 5, 7, 9\}}$
$\boldsymbol{B = \{1, 4, 7, 10\}}$.
then verify the following questions.

(i) $\boldsymbol{A - B = A \cap B'}$

Now $B' = U - B$
$= \{1, 2, 3, 4, ..., 10\} - \{1, 4, 7, 10\}$
$= \{2, 3, 5, 6, 8, 9\}$
L.H.S $= A - B$
$= \{1, 3, 5, 7, 9\} - \{1, 4, 7, 10\}$
$= \{3, 5, 9\}$ (D)

R.H.S $A \cap B'$

$\{1, 3, 5, 7, 9\} \cap \{2, 3, 5, 6, 8, 9\}$

$\{3, 5, 9\}$ (E)

From (D), (E)

$A \quad B \quad A \cap B'$

(ii) $B - A = B \cap A'$

Solution: To show $B \quad A \quad B \cap A'$

L.H.S $\{1, 4, 7, 10\} \quad \{1, 3, 5, 7, 9\}$

$\{4, 10\}$ (R)

Now $A' \quad U \quad A$

$\{1, 2, 3, \ldots, 10\} \quad \{1, 3, 5, 7, 9\}$

$\{2, 4, 6, 8, 10\}$

and R.H.S $B \cap A'$

$\{1, 4, 7, 10\} \cap \{2, 4, 6, 8, 10\}$

$\{4, 10\}$ (Q)

From (R), (Q)

$B \quad A \quad B \cap A'$

(iii) $(A \cup B)' = A' \cap B'$

Solution: To show, $(A \cup B)' \quad A' \cap B'$

L.H.S

$A \cup B$

$\{1,3,5,7,9\} \cup \{1, 4, 7, 10\}$

$\{1, 3, 4, 5, 7, 9, 10\}$ (i)

$(A \cup B)'$

$U \quad (A \cup B)$

$\{1,2,3,\ldots,10\} \quad \{1,3,4,5,7,9,10\}$ from (i)

$\{2, 6, 8\}$ (M)

R.H.S.

$A' \cap B'$

Now $A' \quad \{1, 2, 3, \ldots, 10\} \quad A$

$\{1, 2, 3, \ldots, 10\} - \{1, 3, 5, 7, 9\}$
$\{2, 4, 6, 8, 10\}$
and B' U B
$\{1, 2, 3, \ldots, 10\}$ $\{1, 4, 7, 10\}$
$\{2, 3, 5, 6, 8, 9\}$
Now $A' \cap B'$
$\{2, 4, 6, 8, 10\} \cap \{2, 3, 5, 6, 8, 9\}$
$\{2, 6, 8\}$ (N)
$(A \cup B)'$ $A' \cap B'$

(iv) $(A \cap B)' = A' \cup B'$

Solution: To show $(A \cap B)'$ $A' \cup B'$

Taking L.H.S.
$A \cap B$ $\{1, 3, 5, 7, 9\} \cap \{1, 4, 7, 10\}$
$\{1, 7\}$ (i)
$(A \cap B)'$ U $(A \cap B)$
$\{1, 2, 3, \ldots, 10\}$ $\{1, 7, \}$ from (i)
$\{2, 3, 4, 5, 6, 8, 9, 10\}$ **(P)**

Taking R.H.S.
A' U A
$\{1, 2, 3, \ldots, 10\}$ $\{1, 3, 5, 7, 9\}$
$\{2, 4, 6, 8, 10\}$
B' U B
$\{1, 2, 3, \ldots, 10\} - \{1, 4, 7, 10\}$
$\{2, 3, 5, 6, 8, 9\}$
$A' \cup B'$ $\{2, 4, 6, 8, 10\} \cup \{2, 3, 5, 6, 8, 9\}$
$\{2, 3, 4, 5, 6, 8, 9, 10\}$ **(Q)**

From (P), (Q)
$(A \cap B)'$ $A' \cup B'$

(v) $(A - B)' = A' \cup B$

Solution: To show that $(A$ $B)' = A' \cup B$

Taking L.H.S.

$A - B = \{1, 3, 5, 7, 9\} - \{1, 4, 7, 10\}$
$= \{3, 5, 9\}$

$(A - B)' = U - (A - B)$
$= \{1, 2, 3, \ldots, 10\} - \{3, 5, 9\}$
$= \{1, 2, 4, 6, 7, 8, 10\}$ (S)

Taking R.H.S.

$A' = U - A$
$= \{1, 2, 3, \ldots, 10\} - \{1, 3, 5, 7, 9\}$
$= \{2, 4, 6, 8, 10\}$

Now $A' \cup B$
$= \{2, 4, 6, 8, 10\} \cup \{1, 4, 7, 10\}$
$= \{1, 2, 4, 6, 7, 8, 10\}$ (T)

From (S), (T)

$(A - B)' = A' \cup B$

(vi) $(B - A)' = B' \cup A$

Solution: To show $(B - A)' = B' \cup A$

Taking L.H.S.

$B - A = \{1, 4, 7, 10\} - \{1, 3, 5, 7, 9\}$
$= \{4, 10\}$

$(B - A)' = U - (B - A)$
$= \{1, 2, 3, \ldots, 10\} - \{4, 10\}$
$= \{1, 2, 3, 5, 6, 7, 8, 9\}$ (L)

Taking R.H.S.

$B' = U - B$
$= \{1, 2, 3, \ldots, 10\} - \{1, 4, 7, 10\}$
$= \{2, 3, 5, 6, 8, 9\}$

$B' \cup A = \{2, 3, 5, 6, 8, 9\} \cup \{1, 3, 5, 7, 9\}$
$= \{1, 2, 3, 5, 6, 7, 8, 9\} = \text{L.H.S} = (B - A)'$

2. **If** $U = \{1, 2, 3, 4, \ldots, 10\}$
$A = \{1, 3, 5, 7, 9\}$; $B = \{1, 4, 7, 10\}$;

$C = \{1, 5, 8, 10\}$

then verify the following:

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

Solution:

L.H.S. $= (A \cup B) \cup C$

$= [\{1, 3, 5, 7, 9\} \cup \{1, 4, 7, 10\}] \cup \{1, 5, 8, 10\}$

$= \{1, 3, 4, 5, 7, 9, 10\} \cup \{1, 5, 8, 10\}$

$= \{1, 3, 4, 5, 7, 8, 9, 10\}$ (i)

R.H.S. $= A \cup (B \cup C)$

$= [\{1,3,5,7,9\} \cup [\{1,4,7,10\}] \cup \{1,5,8,10\}]$

$= \{1, 3, 5, 7, 9\} \cup \{1, 4, 5, 7, 8, 10\}$

$= \{1, 3, 4, 5, 7, 8, 9, 10\}$ (ii)

From (i), (ii)

$(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

Solution:

Taking L.H.S.

$(A \cap B) \cap C$

$= [\{1, 3, 5, 7, 9\} \cap \{1, 4, 7, 10\}] \cap \{1, 5, 8, 10\}$

$= \{1, 7\} \cap \{1, 5, 8, 10\}$

$= \{1\}$ (i)

Taking R.H.S.

$A \cap (B \cap C) = \{1, 3, 5, 7, 9\} \cap [\{1, 4, 7, 10\} \cap \{1, 5, 8, 10\}]$

$= \{1, 3, 5, 7, 9\} \cap \{1, 10\}$

$= \{1\}$ (R)

From (i), (ii)

$(A \cap B) \cap C = A \cap (B \cap C)$

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution: Taking L.H.S.

$A \cup (B \cap C)$

$\{1, 3, 5, 7, 9\} \cup [\{1, 4, 7, 10\} \cap \{1, 5, 8, 10\}]$

$\{1, 3, 5, 7, 9\} \cup \{1, 10\}$

$\{1, 3, 5, 7, 9, 10\}$ (R)

R.H.S $(A \cup B) \cap (A \cup C)$

$[\{1,3,5,7,9\} \cup \{1,4,7,10\}] \cap [\{1,3,5,7,9\} \cup \{1,5,8,10\}]$

$\{1, 3, 4, 5, 7, 9, 10\} \cap \{1, 3, 5, 7, 8, 9, 10\}$

$\{1, 3, 5, 7, 9, 10\}$ L.H.S. (Q)

From (R), (Q)

$A \cup (B \cap C)$ $(A \cup B) \cap (A \cup C)$

(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution: Taking L.H.S.

$A \cap (B \cup C)$

$\{1, 3, 5, 7, 9\} \cap [\{1, 4, 7, 10\} \cup \{1, 5, 8, 10\}]$

$\{1, 3, 5, 7, 9\} \cap \{1, 4, 5, 7, 8, 10\}$

$\{1, 5, 7,\}$ (P)

Taking R.H.S. $(A \cap B) \cup (A \cap C)$

$[\{1,3,5,7,9\} \cap \{1,4,7,10\}] \cap [\{1,3,5,7,9\} \cap \{1,5,8,10\}]$

$\{1, 7\} \cup \{1, 5,\}$

$\{1, 5, 7\}$ (Q)

From (P), (Q)

$A \cap (B \cup C)$ $(A \cap B) \cup (A \cap C)$

3. **If $U = N$; then verify De-Morgan's law by using $A = \phi$ and $B = P$.**

Solution: De-Morgan's Law

(i) $(A \cup B)' = A' \cap B'$

Here U N $\{1, 2, 3, \ldots\}$

A ϕ

B $P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$

Let us prove that:

$(A \cup B)'$ $A' \cap B'$

Taking L.H.S.

$A \cup B = \{\ \} \cup \{2, 3, 5, 7, 11, 13 \ldots\ldots\}$

$= \{2, 3, 5, 7, 11, 13 \ldots\ldots\}$

$(A \cup B)' = U - (A \cup B)$

$= \{1, 2, 3, \ldots\} - \{2, 3, 5, 7, 11, 13, 17, \ldots\}$

$= \{1, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18 \ldots\}$ (M)

Taking R.H.S

$A' = U - A$

$= \{1, 2, 3, \ldots\} - \{\ \}$

$= \{1, 2, 3, \ldots\}$

$B' = U - B$

$= \{1, 2, 3, \ldots\} - \{2, 3, 5, 7, 11, 13, 17, \ldots\}$

$= \{1, 4, 6, 8, 9, 10, 12, 14, 15, 16, \ldots\ldots\}$

Now,

$A' \cap B' = \{1,2,3, \ldots\} \cap \{1,4,6,8,9,10,12,14,15,16 \ldots\}$

$= \{1, 4, 6, 8, 9, 10, 12, 14, 15, 16 \ldots\}$ (N)

From (M), (N)

$(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

Solution: L.H.S

$A \cap B = \{\ \} \cap \{2, 3, 5, 7, 11, 13, 17, \ldots\}$

$= \{\ \}$

$(A \cap B)' = U - (A \cap B)$

$= \{1, 2, 3, \ldots\} - \{\ \}$

$= \{1, 2, 3, 4, \ldots\}$ (R)

Taking R.H.S

Now $A' = U - A$

$= \{1, 2, 3, \ldots\} - \{\ \}$

$= \{1, 2, 3, \ldots\}$

and $B' = U - B$

$= \{1, 2, 3, \ldots\} - \{2, 3, 5, 7, 11, 13, 17, \ldots\}$

$= \{1, 4, 6, 8, 9, 10, 12, \ldots\}$

Now

$A' \cup B' = \{1, 2, 3, \ldots\} \cup \{1, 4, 6, 8, 9, 10, 12\}$

$= \{1, 2, 3, 4, ...\}$ (Q)

From R, Q

$(A \cap B)' = A' \cup B'$

4. If $U = \{1, 2, 3, 4, ..., 10\}$, $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 4, 5, 8\}$, then prove the following questions by Venn diagram:

Solution: To show: $A - B = A \cap B'$

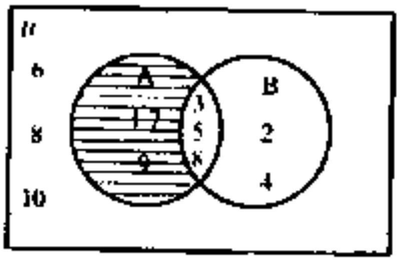


Fig (i)

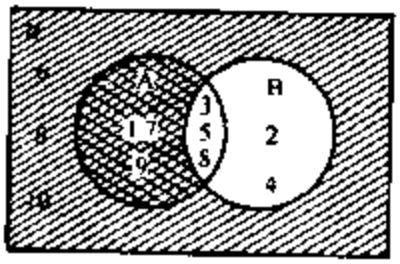


Fig (ii)

$A - B =$ $B' =$ operation (1)

$A \cap B' =$ operation (2)

In fig(i) horizontal line's area and in fig(ii) double shaded area are the same.

(ii) $U = \{1, 2, 3, 4, ..., 10\}$

$A = \{1, 3, 5, 7, 9\}$

$B = \{2, 3, 4, 5, 8\}$

To show $B - A = B \cap A'$

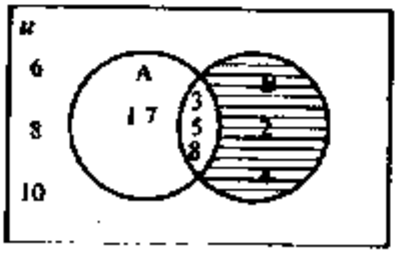


Fig (i)

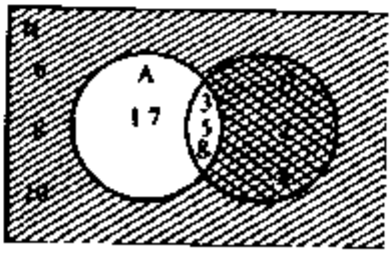
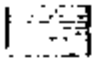


Fig (ii)

B ∩ A =  A′ =  operation (1)

B ∩ A′ =  operation (2)

In fig(i) horizontal lines area and in fig(ii) double shaded area are the same

(iii) $U = \{1, 2, 3, \ldots, 10\}$

$A = \{1, 3, 5, 7, 9\}$

$B = \{2, 3, 4, 5, 8\}$

To show $(B \cup A)' = A' \cap B'$

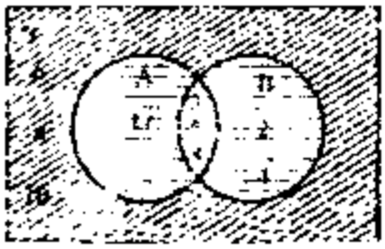


Fig (i)

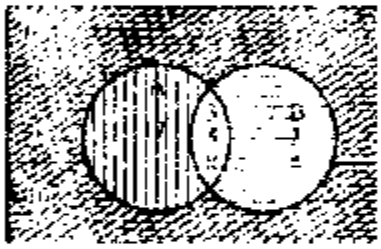


Fig (ii)

A ∪ B =  operation (1)

A′ =  operation (1)

(A ∪ B)′ =  operation (2)

B′ =  operation (2)

$A' \cap B'$ =  operation (3)

In fig(i) slanting line area in $(A \cup B)' = \{6, 8, 10\}$.

In fig(ii) area shown by  is $A' \cap B' = \{6, 8, 10\}$

(iv) $(A \cap B)' = A' \cup B'$

$U = \{1, 2, 3, \ldots, 10\}$

$A = \{1, 3, 5, 7, 9\}$
$B = \{2, 3, 4, 5, 8\}$

To show $(A \cap B)' = A' \cap B'$

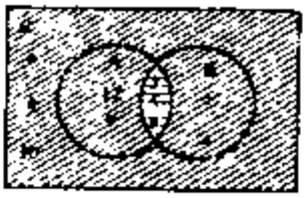


Fig (i)

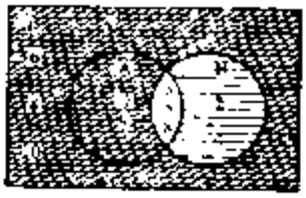
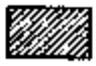


Fig (ii)

$A \cap B =$  operation (1)

A'  operation (1)

$(A \cap B)' =$  operation (2)

(slanting lines area)

$B' =$  operation (2)

$A' \cup B' =$ 

(slanting lines area)

(v) $(A - B)' = A' \cup B'$

$U = \{1, 2, 3, \ldots, 10\}$
$A = \{1, 3, 5, 7, 9\}$
$B = \{2, 3, 4, 5, 8\}$

To show $(A - B)' = A' \cap B$

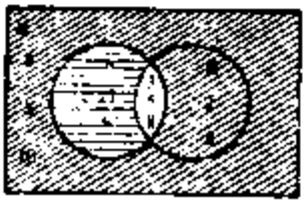
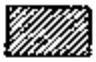


Fig (i)

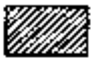

Fig (ii)

$A - B =$  operation (1)

$A' =$  operation (1)

$(A - B)' =$  operation (2)

slanting lines area is $(A - B)'$

$A' \cup B$  operation (2)

slanting lines area is $A' \cup B$

Area in both the cases is the same.

Thus, $(A - B) = A' \cup B$

(vi) $U = \{1, 2, 3, \ldots, 10\}$

$A = \{1, 3, 5, 7, 9\}$

$B = \{2, 3, 4, 5, 8\}$

To show $(B - A)' = B' \cap A$

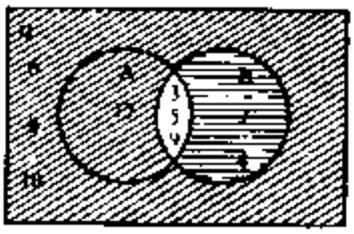


Fig (i)

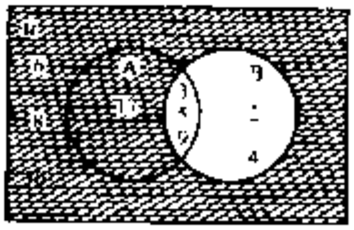


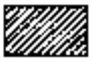
Fig (ii)

$B - A$  operation (1)

B'  operation (1)

$(B - A)'$  operation (2)

(slanting lines area)

$B' \cup A$  operation (2)

(slanting lines area)

Fig(i), (ii) show that $(B - A)' = B' \cup A$.

EXERCISE 5.4

1. If $A = \{a, b\}$ and $B = \{c, d\}$, then find $A \times B$ and $B \times A$.

Solution:
$A = \{a, b\}$
$B = \{c, d\}$

Then $A \times B = \{a, b\} \times \{c, d\}$
$= \{(a, c)(a, d)(b, c)(b, d)\}$

and $B \times A = \{c, d\} \times \{a, b\}$
$= \{(c, a), (c, b), (d, a), (d, b)\}$

2. **If $A = \{0, 2, 4\}$, $B = \{-1, 3\}$, then find $A \times B$, $B \times A$, $A \times A$, $B \times B$.**

Solution:
$A = \{0, 2, 4\}$
$B = \{-1, 3\}$

Then $A \times B = \{0, 2, 4\} \times \{-1, 3\}$
$= \{(0,-1), (0,3), (2,-1), (2,3), (4,-1), (4,3)\}$

(ii) $B \times A = \{-1, 3\} \times \{0, 2, 4\}$
$= \{(-1,0), (-1,2), (-1,4), (3,0), (3,2), (3,4)\}$

(iii) $A \times A = \{0, 2, 4\} \times \{0, 2, 4\}$
$= \{(0,0), (0,2), (0,4), (2,0), (2,2), (2,4), (4,0), (4,2), (4,4)\}$

(iv) $B \times B = \{-1, 3\} \times \{-1, 3\}$
$= \{(-1, -1), (-1,3), (3, -1), (3,3)\}$

3. **Find a and b, if**

(i) $(a - 4, b - 2) = (2, 1)$

$a - 4 = 2$ and $b - 2 = 1$
$a = 2 + 4$ $b = 1 + 2$
$a = 6$ $b = 3$

(ii) $(2a + 5, 3) = (7, b - 4)$

$2a + 5 = 7$ and $3 = b - 4$

$$2a = 7 - 5 \qquad\qquad b = 3 + 4$$
$$2a = 2 \qquad\qquad b = 7$$
$$a = \frac{2}{2}$$
$$a = 1$$

(iii) $(3 - 2a, b - 1) = (a - 7, 2b + 5)$

$$3 - 2a = a - 7 \quad \text{and} \quad b - 1 = 2b + 5$$
$$-2a - a = -7 - 3 \qquad b - 2b = 5 + 1$$
$$-3a = -10 \qquad -b = 6$$
$$a = \frac{10}{3} \qquad b = -6$$

4. Find the sets X and Y, if $X \times Y = \{(a,a),(b,a),(c,a),(d,a)\}$

Solution:

$X \times Y = \{(a, a)(b, a)(c, a)(d, a)\}$

$X = \{a, b, c, d\}$ (second elements of ordered pairs)

$Y = \{a\}$ (second elements of ordered pairs)

5. If $X = \{a, b, c\}$ and $Y = \{d, e\}$, then find the number of elements in

(i) $X \times Y$

Number of elements in $X = 3$

Number of elements in $Y = 2$

Number of elements in $X \times Y = 3 \times 2 = 6$

(ii) $Y \times X$

Number of elements in $Y = 2$

Number of elements in $X = 3$

Number of elements in $Y \times X = 2 \times 3 = 6$

(iii) $X \times X$

Number of elements in $X = 3$

Number of elements in $X \times X = 3 \times 3 = 9$

EXERCISE 5.5

1. If $L = \{a, b, c\}$, $M = \{3, 4\}$, then find two binary relations of $L \times M$ and $M \times L$.

Solution: $L = \{a, b, c\}$, $M = \{3, 4\}$

$L \times M = \{a, b, c\} \times \{3, 4\}$

$= \{(a, 3), (a, 4), (b, 3), (b, 4), (c, 3), (c, 4)\}$

Then $R_1 = \{(a, 3), (b, 4), (c, 3)\}$

$R_2 = \{(a, 4), (b, 3), (c, 4)\}$

Now $M \times L = \{3, 4\} \times \{a, b, c\}$

$= \{(3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\}$

Here $R_1 = \{(3, a), (4, a), (4, c)\}$

$R_2 = \{(3, b), (4, c)\}$

2. If $Y = \{-2, 1, 2\}$, then make two binary relations for $Y \times Y$. Also find their domain and range.

Solution: $Y = \{-2, 1, 2\}$

$Y \times Y = \{-2, 1, 2\} \times \{-2, 1, 2\}$

$= \{(-2, -2), (-2, 1), (-2, 2), (1, -2), (1, 1), (1, 2), (2, -2), (2, 1), (2, 2)\}$

$R_1 = \{(-2, -2), (-2, 1), (1, 2), (2, 2)\}$

Dom $R_1 = \{-2, 1, 2\}$

Range $R_1 = \{-2, 1, 2\}$

and $R_2 = \{(-2, 1), (1, 1), (-2, 2)\}$

Dom $R_2 = \{-2, 1\}$

Range $R_2 = \{1, 2\}$

3. If $L = \{a, b, c\}$, and $M = \{d, e, f, g\}$, then find two binary relations in each:

(i) $L \times L$

$L = \{a, b, c\}$

Now $L \times L = \{a, b, c\} \times \{a, b, c\}$
$= \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$
$R_1 = \{(a, a)\}$
$R_2 = \{(a, b), (b,b), (c, b)\}$

(ii) $L \times M$
$L = \{a, b, c\}$
$M = \{d, e, f, g\}$
Now $L \times M = \{a, b, c\} \times \{d, e, f, g\}$
$= \{(a,d), (a,e), (a,f), (a,g), (b,d), (b,e),$
$(b,f), (b,g), (c,d), (c,e), (c,f), (c,g)\}$
Here, $R_1 = \{(a, d), (b, f)\}$
$R_2 = \{(b, e), (b,g), (c, d), (c, e)\}$

(iii) $M \times M$
$M = \{d, e, f, g\}$
Now $M \times M = \{d, e, f, g\} \times \{d, e, f, g\}$
$= \{(d,d), (d,e), (d,f), (d,g), (e,d), (e,e), (e,f), (e,g),$
$(f,d), (f,e), (f,f), (f,g), (g,d), (g,e), (g,f), (g,g)\}$
Here, $R_1 = \{(d, e), (d, f), (f, f)\}$
$R_2 = \{(d, f), (e, d), (e, e), (g, g)\}$

4. ***If set M has 5 elements, then find the number of binary relations in M.***

Solution: Number of elements in $M = 5$
Number of elements in $M = 5$
Number of binary relations in $M \times M = 2^{5 \times 5} = 2^{25}$

5. **If $L = \{x \mid x \in N \wedge x \leq 5\}$, $M = \{y \mid y \in P \wedge y < 10\}$, then make the following relations from L to M**

(i) $R_1 = \{(x, y) \mid y < x\}$

Solution: $L = \{x \mid x \in N \wedge x \leq 5\}$
Thus, $L = \{1, 2, 3, 4, 5\}$
and $M = \{y \mid y \in P \wedge y < 10\}$
Thus, $M = \{2, 3, 5, 7\}$

Now, $L \times M$ = {1, 2, 3, 4, 5}, {2, 3, 5, 7}
= {(1,2), (1,3), (1,5), (1,7),(2,2), (2,3), (2, 5),
(2, 7), (3,2), (3,3), (3,5), (3,7), (4,2), (4,3),
(4,5), (4,7), (5,2), (5,3), (5,5), (5,7)}

R_1 = {(x, y)| y < x}
= {(3,2), (4,2), (4,3), (5,2), (5,3) }

Dom R_1 = {3, 4, 5},
Range R_1 = {2, 3}

(ii) R_2 = {(x, y)| y = x}
R_2 = {(2, 2), (3, 3), (5, 5)}
Dom R_2 = {2, 3, 5}
Range R_2 = {2, 3, 5}

(iii) R_3 = {(x, y)| y + x = 6}
R_3 = {(1, 5), (3, 3), (4, 2)}
Dom R_3 = {1, 3, 4},
Range R_3 = {5, 3, 2}

(iv) R_4 = {(x, y)| y − x = 2}
R_4 = {(1, 3), (3, 5), (5, 7)}
Dom R_4 = {1, 3, 5},
Range R_4 = {3, 5, 7}

Also write the domain and range of each relation.

6. ***Indicate relations, into function, one-one function, onto function, and bijective function from the following. Also find their domain and the range.***

(i) R_1 = {(1, 1), (2, 2), (3, 3),(4, 4)}
Bijective
Dom R_1 = {1, 2, 3, 4}
Range R_1 = {1, 2, 3, 4}

(ii) R_2 = {(1, 2), (2, 1), (3, 4),(3, 5)}
Relation
Dom R_2 = {1, 2, 3}
Range R_2 = {1, 2, 4, 5}

(iii) $R_3 = \{(b, a), (c, a), (d, a)\}$

Function

Dom R_3 $\{b, c, d\}$

Range R_3 $\{a\}$

(iv) $R_4 = \{(1, 1), (2, 3), (3, 4), (4, 3), (5, 4)\}$

Onto function

Dom R_4 $\{1, 2, 3, 4, 5\}$

Range R_4 $\{1, 3, 4\}$

(v) $R_5 = \{(a, b), (b, a), (c, d), (d, e)\}$

One-one function

Dom R_5 $\{a, b, c, d\}$

Range R_5 $\{a, b, d, e\}$

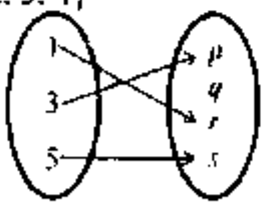
(vi) $R_6 = \{(1, 2), (2, 3), (1, 3), (3, 4)\}$

Relation

Dom R_6 $\{1, 2, 3\}$

Range R_6 $\{2, 3, 4\}$

(vii) $R_7 =$

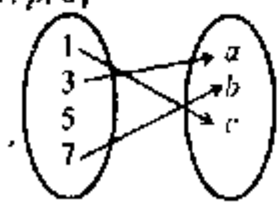


one – one function

Dom R_7 $\{1, 2, 3\}$

Range R_7 $\{r, p, s\}$

(viii) $R_8 =$



Relation

Dom R_8 $\{1, 3, 7\}$

Range R_8 $\{c, a, b\}$

EXERCISE 6.1

1. ***The following data shows the number of members in various families. Construct frequency distribution. Also find cumulative frequencies.***

9, 11, 4, 5, 6, 8, 4, 3, 7, 8, 5, 5, 8, 3, 4, 9, 12, 8, 9, 10, 6, 7, 7, 11, 4, 4, 8, 4, 3, 2, 7, 9, 10, 9, 7, 6, 9, 5, 7.

Solution:

Frequency distribution of number of family members.

Number of members	Talley marks	Frequency	Commutative frequency
2	I	1	1
3	III	3	1 + 3 = 4
4	IIII I	6	4 + 6 = 10
5	IIII	4	10 + 4 = 14
6	III	3	14 + 3 = 17
7	IIII I	6	17 + 6 = 23
8	IIII	5	23 + 5 = 28
9	IIII I	6	28 + 6 = 34
10	II	2	34 + 2 = 36
11	II	2	36 + 2 = 38
12	I	1	38 + 1 = 39
	Total	39	

2. ***The following data has been obtained after weighing 40 students of class V. Make a frequency distribution taking class interval size as 5. Also find the class boundaries and midpoints.***

34, 26, 33, 32, 24, 21, 37, 40, 41, 28, 28, 31, 33, 34, 37, 23, 27, 31, 31, 36, 29, 35, 36, 37, 38, 22, 27, 28, 29, 31, 35, 35, 40, 21, 32, 33, 27, 29, 30, 23.

Also make a less than cumulative frequency distribution. (Hint: Make classes 20 – 24, 25 – 29…..).

Solution:

Frequency Distribution

Class Limits	Talley marks	Frequency
20 - 24	IIII I	6
25 - 29	IIII IIII	10
30 - 34	IIII IIIII	12
35 - 39	IIIIIII	9
40 - 44	III	3
Total		40

Cumulative Frequency Distribution

Class Boundaries	*Frequency f*	*Commulative Frequency*	*Class Boundaries*	*Commulative Frequency*
14.5 - 19.5	0	0	Less than 19.5	0
19.5 - 24.5	6	0 + 6 = 6	Less than 24.5	6
24.5 - 29.5	10	6 + 10 = 16	Less than 29.5	16
29.5 - 34.5	13	16 + 13 = 29	Less than 34.5	29
34.5 - 39.5	8	29 + 8 = 37	Less than 39.5	37
39.5 - 44.5	3	37 + 3 = 40	Less than 44.5	40

3. *From the following data representing the salaries of 30 teachers of a school. Make a frequency distribution taking class interval size of* **Rs. 100. 450, 500, 550, 580, 670, 1200, 1150, 1120, 950, 1130, 1230, 890,**

780, 760, 670, 880, 890, 1050, 980, 970, 1130, 1220,

760, 690, 710, 750, 1170, 760, 1240.

(Hint: Make classes 450 – 549, 550 – 649,......).

Frequency Distribution Table

Class Limits	Tallies marks	Frequency
450 – 549	II	2
550 – 649	II	2
650 – 749	IIII	4
750 – 849	IIII	5
850 – 949	III	3
950 – 1049	IIII	4
1050 – 1149	IIII	5
1150 – 1249	IIII	5
	Total	30

4. ***The following data shows the daily load shedding duration in hours in 30 localities of a certain city. Make a frequency distribution of the load shedding duration taking 2 hours as class interval size and answer the following questions.***

6, 12, 5, 7, 8, 3, 6, 7, 10, 2, 14, 11, 12, 8, 6, 8, 9, 7, 11, 3, 9, 12, 13, 10, 14, 7, 6, 10, 11, 14, 12.

(a) Find the most frequent load shedding hours?

(b) Find the least load shedding intervals?

(Hint: Make classes 2 – 3, 4 – 5, 6 – 7.....)

Frequency Distribution Table

Class Limits	Tallies marks	Frequency
2 – 3	II	2
4 – 5	I	1
6 – 7	IIII IIII	9
8 – 9	IIII	5

10 - 11	IIIII	6
12 - 13	IIII	5
14 - 15	III	3
	Total	31

Q.(a) Find the most frequent load shedding hours.

A. 6 - 7

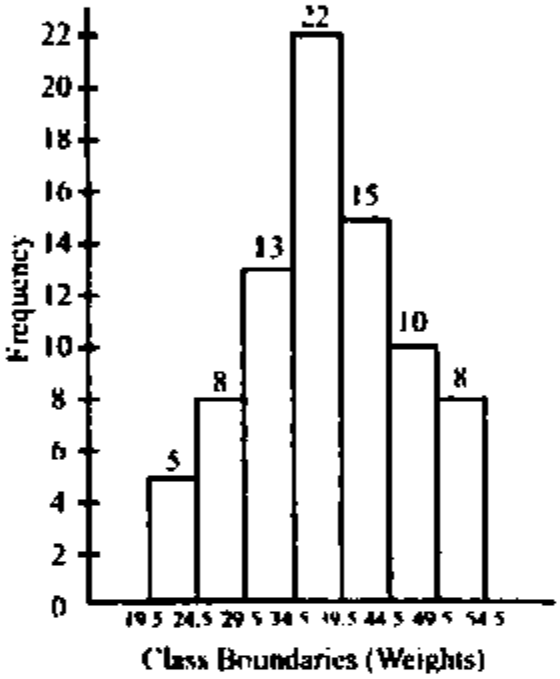
Q.(b) Find the least load shedding intervals.

A. 4 – 5

5. ***Construct a Histogram and frequency Polygon for the following data showing weights of students in kg.***

Weights	***Frequency / No. of Students***
20 – 24	5
25 – 29	8
30 - 34	13
35 - 39	22
40 – 44	15
45 - 49	10
50 – 54	8

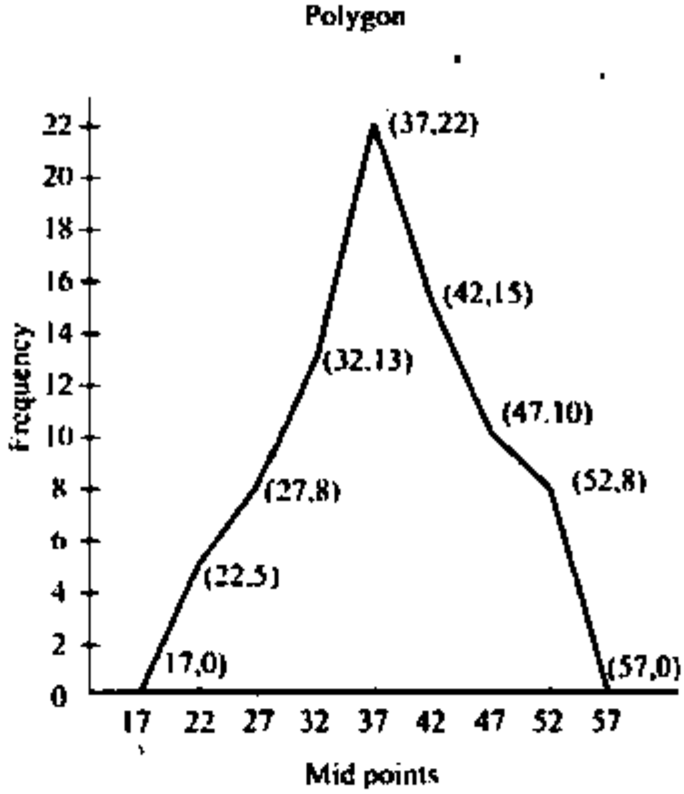
Class Boundaries	Frequency
19.5 - 24.5	5
24.5 - 29.5	8
29.5 - 34.5	13
34.5 - 39.5	22
39.5 - 44.5	15
44.5 – 49.5	10
49.5 - 54.5	8



Construction of frequency polygon

We take two additional groups with the same class limit.

Class Limits		*Class Boundaries*		*Mid points*	*Frequency*
				17	0
20	24	19.5	24.5	22	5
25	29	24.5	29.5	27	8
30	34	29.5	34.5	32	13
35	39	34.5	39.5	37	22
40	44	39.5	44.5	42	15
45	49	44.5	49.5	47	10
50	54	49.5	54.5	52	8
55	59	54.5	59.5	57	0



EXERCISE 6.2

1. ***What do you understand by measures of central tendency.***

Ans. The specific value of the variable around which the majority of the observations tend to concentrate is called the central tendency.

2. ***Define (i) Arithmetic mean (ii) Geometric mean, (iii) Harmonic mean (iv) mode (v) median.***

Ans. Mean is a measure that determines a value of the variable under study by dividing the sum of all values of the variable by their number of observations.

(ii) Geometric Mean:

Geometric mean of a variable x is the nth positive root of the product of the $x_1, x_2, x_3, \ldots xn$ observations.

$G.M = (x_1 \times x_2 \times x_3 \ldots . xn)^{1/n}$

(iii) Harmonic Mean:

Harmonic mean refers to the value obtained by reciprocating the mean of the reciprocal of $x_1, x_2, x_3, \ldots xn$ observations.

$$H.M = \frac{n}{\Sigma \frac{1}{x}}$$

$$H.M = \frac{n}{\Sigma \frac{f}{x}}$$

(iv) Mode:

The most repeated value in an observation is called its *mode*.

(v) Median:

Median is the middle most observation in an arranged data set. It divides the data set into two equal parts.

3. Find arithmetic mean by direct method for the following set of data:

(i) 12, 14, 17, 20, 24, 29, 35, 45.

Solution:

$$A.M = \bar{X} = \frac{\Sigma x}{n} = \frac{12+14+17+20+24+29+35+45}{8}$$

$$= \frac{196}{8}$$

$$= 24.5$$

(ii) 200, 225, 350, 375, 270, 320, 290.

$$A.M = \bar{X} = \frac{\Sigma x}{n} = \frac{200+225+350+375+270+320+290}{7}$$

$$= \frac{2030}{7}$$

$$= 290$$

4. ***For each of the data in Q. no 3., compute arithmetic mean using indirect method.***

(i) ***12, 14, 17, 20, 24, 29, 35, 45.***

Solution: Take any constant say 24 and take deviations from it (24).

X	D = X − A
12	12 − 24 = −12
14	14 − 24 = −10
17	17 − 24 = −7
20	20 − 24 = −4
24	24 − 24 = 0
29	29 − 24 = 5
35	35 − 24 = 11
45	45 − 24 = 21
$n = 8$	$\Sigma D = 4$

$$\bar{X} = A + \frac{\Sigma D}{n}$$

$$= 24 + \frac{4}{8}$$

$$= 24 + \frac{1}{2}$$

$$= 24\frac{1}{2} = 24.5$$

(ii) ***200, 225, 350, 270, 320, 290***

Take any constant 270 and take deviations from it (270)

$A = 270$

X	$D = X - A$
200	$200 - 270 = -70$
225	$225 - 270 = -45$
350	$350 - 270 = 80$
375	$375 - 270 = 105$
270	$270 - 270 = 0$
320	$320 - 270 = 50$
290	$290 - 270 = 20$
$n = 7$	$\Sigma D = 140$

$\Sigma D = 140$

$N = 7$

$$\bar{X} = A + \frac{\Sigma D}{n}$$

$$= 270 + \frac{140}{7}$$

$\bar{X} = 270 + 20 = 290$

5. ***The marks obtained by students of class XI in mathematics are given below. Compute arithmetic mean by direct and indirect methods.***

Classes / Groups	*Frequency*
0 – 9	2
10 – 19	10
20 – 29	5
30 – 39	9
40 – 49	6
50 – 59	7
60 – 69	1

Direct Method:

Classes/groups	mid-points (x)	f	$f(x)$
0 – 9	4.5	2	$4.5\times = 9.0$
10 – 19	14.5	10	$14.5\times 10 = 145.0$
20 – 29	24.5	5	$24.5\times 5 = 122.5$
30 – 39	34.5	9	$34.5\times 9 = 310.5$
40 – 49	44.5	6	$44.5\times 6 = 267.0$
50 – 59	54.5	7	$54.5\times 7 = 381.5$
60 – 69	64.5	1	$64.5\times 1 = 64.5$
		$n=\Sigma f = 40$	1300

$$\overline{X} = \frac{\Sigma fx}{\Sigma f} = \frac{1300}{40}$$

$$= 32.5$$

Indirect, short cut method

Let A = 34.5

Classes / groups	f	mid-point (x)	$D = X - A$	$U = \frac{D}{10}$	fD	$f(U)$ $\frac{f(D)}{10}$
0 - 9	2	4.5	$4.5 - 34.5 = -30$	–3	–60	–6
10 – 19	10	14.5	$14.5 - 34.5 = -20$	–2	–200	–20
20 – 29	5	24.5	$24.5 - 34.5 = -10$	–1	–50	–5
30 - 39	9	34.5	$34.5 - 34.5 = 0$	0	0	0
40 – 49	6	44.5	$44.5 - 34.5 = 10$	1	60	6
50 - 59	7	54.5	$54.5 - 34.5 = 20$	2	140	14
60 – 69	1	64.5	$64.5 - 34.5 = 30$	3	30	3
Total	40				–80	–8

$$\overline{X} = A + \frac{\Sigma fD}{\Sigma f} \text{ or using: } \overline{X} = A + \frac{\Sigma f(U)}{\Sigma f}\times h$$

$$34.5 + \frac{(-80)}{40} \qquad\qquad = 34.5 + \frac{-8}{40} \times h$$

$$34.5 - 2 \qquad\qquad = 34.5 + \frac{-8}{40} \times 10$$

$$32.5 \qquad\qquad = 34.5 - 2$$

$$= 32.5$$

The following data relates to the age of children in a school. Compute the mean age by direct and short-cut method taking any provisional mean. (Hint. Take A = 8)

Class limits	***Frequency***
4 – 6	10
7 – 9	20
10 – 12	13
13 – 15	7
Total	50

Also Compute Geometric mean and Harmonic mean.

Solution: Direct Method:

Class Limits	mid points (x)	f	$f(x)$
4 – 6	5	10	$5 \times 10 = 50$
7 – 9	8	20	$8 \times 20 = 160$
10 – 12	11	13	$11 \times 13 = 143$
13 – 15	14	7	$14 \times 7 = 98$
Total		$\Sigma f = 50$	$\Sigma f(x) = 451$

$$A.M = \frac{\Sigma f(x)}{\Sigma f} = \frac{451}{50} = 9.02$$

Indirect / short method

Let A = 11

Classes limits	f	mid-point (x)	$D = X - A$	$u = \frac{D}{3}$	fD	$fu = \frac{f(D)}{3}$
4 – 6	10	5	5 – 11 = –6	–2	–60	–20
7 – 9	20	8	8 – 11 = –3	–1	–60	–20
10 – 12	13	11	11 – 11 = 0	0	0	0
13 – 15	7	14	14 – 11 = 3	1	21	7
	50				–99	–33

$$\overline{X} = A + \frac{\Sigma fD}{\Sigma f} \text{ or using } \overline{X} = A + \frac{\Sigma fu}{\Sigma f} \times h$$

$= 11 - \frac{99}{50}$ $\qquad$ $= 11 - \frac{33}{50} \times 3$

$= 11 - 1.98$ $\qquad$ $= 11 - \frac{99}{50}$

$= 9.02$ $\qquad$ $= 11 - 1.98$

$= 9.02$

Geometric Mean:

We proceed as follows:

Class limits	f	mid points x	$\log x$	$f \log x$
4 – 6	10	5	0.69897	6.9897
7 – 9	20	8	0.90309	18.0618
10 – 12	13	11	1.04139	13.53807
13 – 15	7	14	1.14613	8.02291
Σf	50			$\Sigma f \log x = 46.61248$

$$G.M = \text{Anti}\log\left(\frac{\Sigma f \log x}{\Sigma f}\right)$$

$\text{Anti log } \frac{46.61248}{50}$

$\text{Anti log } .9322496$

8.553

Harmonic Mean

Class limits	f	mid points x	$\frac{f}{x}$
4 - 6	10	5	$\frac{10}{5} = 2.0$
7 - 9	20	8	$\frac{20}{8} = 2.5$
10 - 12	13	11	$\frac{13}{11} = 1.18$
13 - 15	7	14	$\frac{7}{14} = 0.50$
	$\Sigma f = 50$		$\Sigma \frac{f}{x} = 6.18$

$$H.M = \frac{\Sigma f}{\Sigma \frac{f}{x}} = \frac{50}{6.18} = 8.09$$

7. ***The following data show the number of children in various families. Find mode and median.***
9, 11, 4, 5, 6, 8, 4, 3, 7, 8, 5, 5, 8, 3, 4, 9, 12, 8, 9, 10, 6, 7, 7, 11, 4, 4, 8, 4, 3, 2, 7, 9, 10, 9, 7, 6, 9, 5.

Solution: Writing the observations in ascending order.

2, 3, 3, 3, 4, 4, 4, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 9, 9, 9, 9, 9, 9, 10, 10, 11, 11, 12.

Mode: The most frequent observation = 9, 4

Number of observations = 38

Therefore, median is the mean of 19^{th} and 20^{th} observation $\frac{7+7}{2} = 7$

8. ***Find Model number of heads for the following distribution showing the number of heads when 5 coins are tossed. Also determine median.***

X (number of heads)	***Frequency (number of times)***
1	3
2	8
3	5
4	3
5	1
	$n = \Sigma f = 20$

Solution: Mode: The most frequent observation = 2

For median, we make cumulative frequency column.

X	Frequency	Cumulative frequency
1	3	3
2	8	3 + 8 = 11
3	5	11 + 5 = 16
4	3	16 + 3 = 19
5	1	19 + 1 = 20

Median = the class containing $\left(\frac{n}{2}\right)^{th}$ observation.

= the class containing $\left(\frac{20}{2}\right)^{th}$ observation.

= the class containing $(10)^{th}$ observations.

Median = 2

9. ***The following frequency distribution the weights of boys in kilogram. Compute mean, median, mode.***

Class Intervals	*Frequency*
1 3	2
4 6	3
7 9	5
10 12	4
13 15	6
16 18	2
19 21	1
	$n = \Sigma f = 23$

Solution:

c.I	f	mid points x	fx	Class boundaries	Cumulative frequency
1 3	2	2	4	0.5 3.5	2
4 6	3	5	15	3.5 6.5	2 + 3 5
7 9	5	8	40	6.5 9.5	5 + 5 10
10 12	4	11	44	9.5 12.5	10 + 4 14
13 15	6	14	84	12.5 15.5	14 + 6 20
16 18	2	17	34	15.5 18.5	20 + 2 22
19 21	1	20	20	18.5 21.5	22 + 1 23
	23		241		

Mean $\bar{X} = \dfrac{\Sigma fx}{\Sigma f}$

$= \dfrac{241}{23}$

$= 10.478$

Median:

Median class = class containing $\left(\dfrac{n}{2}\right)^{th}$ observation

$$=\left(\frac{23}{2}\right)^{th} = (11.5)^{th} \text{ observation.}$$

Median class is 9.5 - 12.5

Here $l = 9.5$

$c = 10$

$f = 4$

$h = 3$

$$\tilde{x} = l + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

$$= 9.5 + \frac{3}{4}\left(\frac{23}{2} - 10\right)$$

$$= 9.5 + \frac{3}{4}\left(\frac{3}{2}\right)$$

$$= 9.5 + \frac{9}{8}$$

$$= 9.5 + 1.125$$

$$= 10.625$$

Mode: $= l + \dfrac{fm - f_1}{2fm - f_1 - f_2} \times h$

Here $l = 12.5$

$fm = 6$

$f_1 = 4$

$f_2 = 2$

$h = 3$

$\therefore$ Mode $= 12.5 + \dfrac{6 - 4}{2(6) - 4 - 2} \times 3$

$$= 12.5 + \frac{2}{6} \times 3$$

$$= 12.5 + 1$$

$$= 13.5$$

10. ***A student obtained the following marks at a certain examination: English 73, Urdu 82, Mathematics 80, History 67 and Science 62.***

(i) **If the weights accorded these marks are 4, 3, 3, 2 and 2, respectively, what is an appropriate average mark?**

(ii) **What is the average mark if equal weights are used?**

Solution:

Marks (x)	Weight w	xw	
73	4	73×4	292
82	3	82×3	246
80	3	80×3	240
67	2	67×2	134
62	2	62×2	124
$\Sigma x = 364$	$\Sigma w = 14$	Σxw	1036

$$\bar{X}_w = \frac{\Sigma xw}{\Sigma w} = \frac{1036}{14} = 74$$

$$\bar{X} = \frac{\Sigma x}{n}$$

$$= \frac{364}{5}$$

$$\bar{X} = 72.8$$

11. ***On a vacation trip a family bought 21.3 liters of petrol at 39.90 rupees per liter, 18.7 liters at 42.90 rupees per liter, and 23.5 liters at 40.90 rupees per liter. Find the mean price paid per liter.***

Litters x	Price per liter, P	Amount xp	
21.3	39.90	(21.3)(39.90)	849.87
18.7	42.90	(18.7)(42.90)	802.23

23.5	40.90	(23.5)(40.90) = 961.15
$\Sigma x = 63.5$		$\Sigma xP = 2613.25$

$$\text{Mean price} = \frac{\Sigma xP}{\Sigma x}$$

$$= \frac{2613.25}{63.5}$$

$$= 41.15 \text{ rupees per liter}$$

12. ***Calculate simple moving average of 3 years from the following data:***

Years	2001	2002	2003	2004	2005	[illegible]	[illegible]	[illegible]	2009	2010
Values	102	108	130	140	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Solution:

Years	Values	3 years moving Total	Average	
2001	102	--	–	Examples
2002	108	340	340/3 = 113.33	102+108+130 = 340
2003	130	378	378/3 = 126.00	108+130+140 = 378
2004	140	428	428/3 = 142.67	130+140+158 = 428
2005	158	478	478/3 = 159.33	
2006	180	534	534/3 = 178.00	
2007	196	586	586/3 = 195.33	
2008	210	626	626/3 = 208.67	
2009	220	660	660/3 = 220.00	
2010	230	–	–	

13. ***Determine graphically for the following data and check your answer by using formulae.***

(i) Median and Quartiles using cumulative frequency polygon.

(ii) Mode using Histogram.

Class Boundaries	*Frequency*
10 – 20	2
20 – 30	5
30 – 40	9
40 – 50	6
50 – 60	4
60 – 70	1

Solution:

Class Boundaries	f	$c.f$	
10 – 20	2	2	
20 – 30	5	7	
30 – 40	9	16	Median class.
40 – 50	6	22	Q_3 class
50 – 60	4	26	
60 – 70	1	27	
	$\Sigma f = 27$		

Median class $= \left(\frac{n}{2}\right)^{th}$ observation.

$= \left(\frac{27}{2}\right)^{th} = (13.5)^{th}$ observation

$$\text{Median} = l + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

Here $l = 30$
$h = 10$
$f = 9$
$n = 27$
$c = 7$

thus median $\tilde{x} = 30 + \frac{10}{9}\left(\frac{27}{2} - 7\right)$

$= 30 + \frac{10}{9}\left(\frac{13}{2}\right)$

$= 30 + \frac{65}{9}$

$= 30 + 7.22$

$= 37.22$

To find Q_3.

We find $3\left(\frac{n}{4}\right)^{th}$ observation.

Q_3. Class $= 3\left(\frac{n}{4}\right)^{th}$ observations.

$= 3\left(\frac{27}{4}\right)^{th}$ observations.

$= 3(6.75)^{th}$ observations.

$= (20.25)^{th}$ observations.

Q_3 Class is $40 - 50$

Now $\quad Q_3 = l + \frac{h}{f}\left(\frac{3n}{4} - c\right)$

Here, $\quad l = 40$

$h = 10$

$f = 6$

$n = 27$

$c = 16$

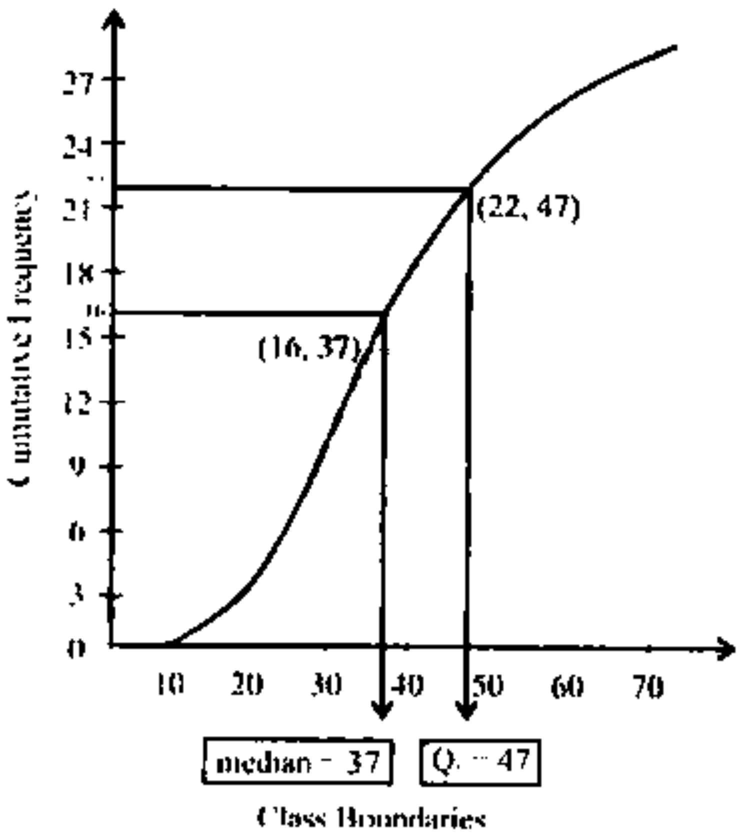
$Q_3 = 40 + \frac{10}{6}\left(\frac{3 \times 27}{4} - 16\right)$

$= 40 + \frac{10}{6}(20.25 - 16)$

$$40 + \frac{10}{6}(4.25)$$

$$40 + 7.08$$

$$47.08$$



Median = 37

Q_3 = 47

EXERCISE 6.3

1. What do you understand by Dispersion?

Ans. Dispersion means the spread or scaterness of observations in a data set. By dispersion we mean the extent to which the observations in a sample or in a population are spread out. The main measures of dispersion are three. Range, variance and standard deviation

2. How do you define measures of dispersion?

Ans. **Range:**

Range measures the extent of variation between two extreme observations of a data set. It is given by the formula Range = $X_{max} - X_{min}$

Range = (upper C.B of the last group) - (lower C.B of first group)

3. Define Range, Standard deviation and Variance.

Ans. **Variance:**

The mean of the squared deviations of $x_i (i = 1, 2, \ldots, n)$

$$\text{Variance} = S^2 = \frac{\Sigma(X - \bar{X})^2}{n}$$

$$= S^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$$

Standard Variance

$$= S = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n}}$$

$$S = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

4. ***The salaries of five teachers in Rupees are as follows. 11500, 12400, 15000, 14500, 14800.***

Find Range and standard deviation.

Solution: X = 11500, 12400, 15000, 14500, 14800

Here, X_{max} = 15000

X_{min} = 11500

Range = Xmax − Xmin

= 15000 − 11500

= 3500

$$\bar{X} = \frac{\Sigma x}{n}$$

$$= \frac{11500+12400+15000+14500+14800}{5}$$

$$= \frac{68200}{5}$$

= 13640

X	$X - \bar{X}$	$(X - \bar{X})^2$
11500	−2140	4579600
12400	−1240	1537600
15000	1360	1849600
14500	860	739600
14800	1160	1345600

$\Sigma x = 68200$ $\Sigma(X - \bar{X})^2 = 10052000$

$n = 5$

$\bar{X} = 13640$

$$\text{S.D} \quad S \quad \sqrt{\frac{\Sigma(X-\overline{X})^2}{n}}$$

$$\sqrt{\frac{10052000}{5}}$$

$$\sqrt{2010400}$$

$$1417.88$$

5. ***a. Find the standard deviation "S" of each set of numbers:***

(i) 12, 6, 7, 3, 15, 10, 18, 5

(ii) 9, 3, 8, 8, 9, 8, 9, 18.

Solution: X 12, 6, 7, 3, 15, 10, 18, 5

X	$X-\overline{X}$	$(X-\overline{X})^2$
12	2.5	6.25
6	−3.5	12.25
7	−2.5	6.25
3	−6.5	42.25
15	5.5	30.25
10	0.5	0.25
18	8.5	72.25
5	−4.5	20.25

$\Sigma x \quad 76 \qquad \Sigma(X - \overline{X})^2 \quad 190$

$n \quad 8$

$$\overline{X} \quad \frac{76}{8}$$

$$\overline{X} \quad 9.5$$

$$S.D = S = \sqrt{\frac{\Sigma(X-\bar{X})^2}{n}}$$
$$= \sqrt{\frac{190}{8}}$$
$$= \sqrt{23.75}$$
$$= 4.87$$

(ii) *9, 3, 8, 8, 9, 8, 9, 18*

X = 9, 3, 8, 8, 9, 8, 9, 18

X	$X-\bar{X}$	$(X-\bar{X})^2$
9	0	0
3	-6	36
8	-1	1
8	-1	1
9	0	0
8	-1	1
9	0	0
18	9	81

$\Sigma x = 72$ $\quad \Sigma(X-\bar{X})^2 = 120$

$n = 8$

$$\bar{X} = \frac{\Sigma x}{n}$$
$$\bar{X} = \frac{72}{8}$$
$$= 9$$

$$S.D = S = \sqrt{\frac{\Sigma(X-\bar{X})^2}{n}}$$
$$= \sqrt{\frac{120}{8}}$$
$$= \sqrt{15}$$

3.87

b. Calculate variance for the data: 10, 8, 9, 7, 5, 12, 8, 6, 8, 2.

Solution: X 10, 8, 9, 7, 5, 12, 8, 6, 8, 2

X	$X-\bar{X}$	$(X-\bar{X})^2$
10	2.5	6.25
8	0.5	.25
9	1.5	2.25
7	− 0.5	.25
5	− 2.5	6.25
12	4.5	20.25
8	0.5	.25
6	− 1.5	2.25
8	0.5	.25
2	− 5.5	30.25

$\Sigma x = 75$ $\qquad \Sigma(X-\bar{X})^2 = 68.5$

$n = 10$

$\bar{X} = \dfrac{\Sigma x}{n}$

$\bar{X} = \dfrac{75}{10}$

$\bar{X} = 7.5$

Variance $S^2 = \dfrac{\Sigma(X-\bar{X})^2}{n}$

$= \dfrac{68.5}{10}$

$= 6.85$

6. *The length of 32 items are given below. Find the mean length and standard deviation of the distribution.*

Length	20 – 22	23 - 25	26 – 28	29 – 31	32 – 34
Frequency	3	6	12	9	2

Solution:

C.I	f	mid point (x)	fx	$x-\overline{X}$	$(X-\overline{x})^2$	$f(X-\overline{X})^2$
20 – 22	3	21	63	– 6	36	108
23 - 25	6	24	144	– 3	9	54
26 – 28	12	27	324	0	0	0
29 – 31	9	30	270	3	9	81
32 – 34	2	33	66	6	36	72
	32	$\Sigma fx = 867$			90	315

$$\overline{X} = \frac{\Sigma fx}{n} = \frac{867}{32} = 27.093 = 27 \text{ approx.}$$

$$\text{S.D} = S^2 = \sqrt{\frac{\Sigma(X-\overline{X})^2}{n}}$$

$$= \sqrt{\frac{315}{32}}$$

$$= \sqrt{9.84375}$$

$$= 3.137$$

7. *For the following distribution of marks calculate Range.*

Marks in percentage	*Frequency / (No of Students)*
33 – 40	28
41 – 50	31
51 – 60	12
61 – 70	9
71 – 75	5
	= 85

Solution:

C.I	Class Boundaries	f
33 - 40	32.5 – 40.5	28
41 - 50	40.5 – 50.5	31
51 - 60	50.5 – 60.5	12
61 - 70	60.5 – 70.5	9
71 - 75	70.5 – 75.5	5

Here, $X_{max} = 75.5$

$X_{min} = 32.5$

$$\text{Range} = X_{max} - X_{min}$$
$$= 75.5 - 32.5$$
$$= 43$$

MISCELLANEOUS EXERCISE – 6

1. **Multiple Choice Questions**

Three possible answers are given for the following questions. Tick (✓) the correct answer.

(i) A grouped frequency table is also called
- (a) data
- (b) frequency distribution
- (c) frequency polygon

(ii) A histogram is a set of adjacent
- (a) square
- (b) rectangles
- (c) circles

(iii) A frequency polygon is a many sided
- (a) closed figure
- (b) rectangle
- (c) square

(iv) A cumulative frequency table is also called
- (a) frequency distribution
- (b) data
- (c) less than cumulative frequency distribution

7 INTRODUCTION TO TRIGONOMETRY

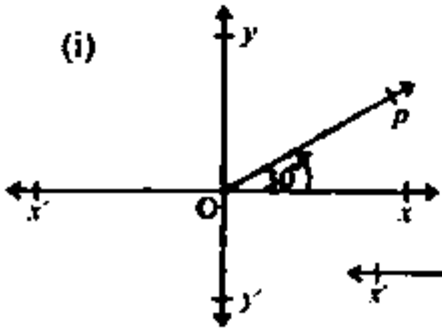
EXERCISE 7.1

Q.1 Locate the following angles.

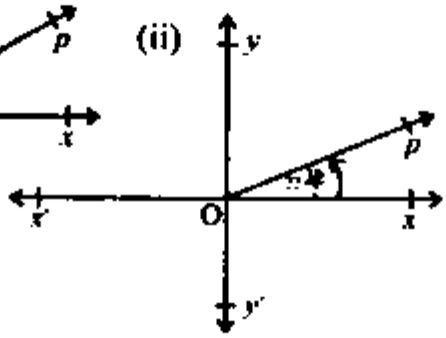
(i) 30° (ii) $22\frac{1}{2}^{\circ}$ (iii) 135° (iv) 225°

(v) -60° (vi) -120° (vii) -150° (viii) -225°

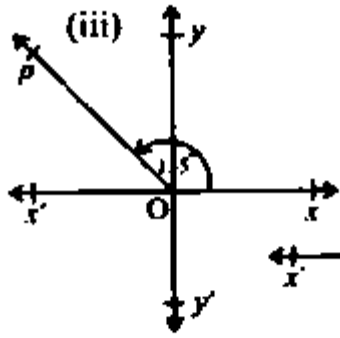
(i)



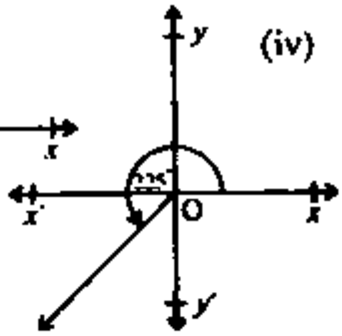
(ii)



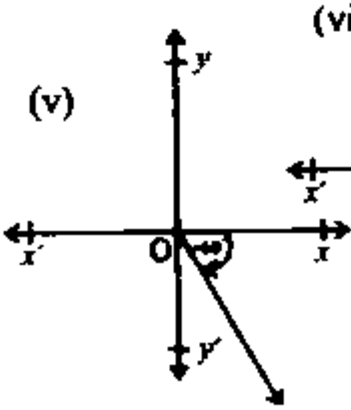
(iii)



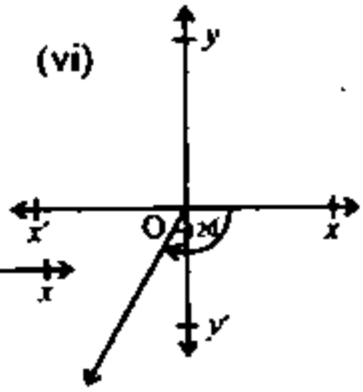
(iv)

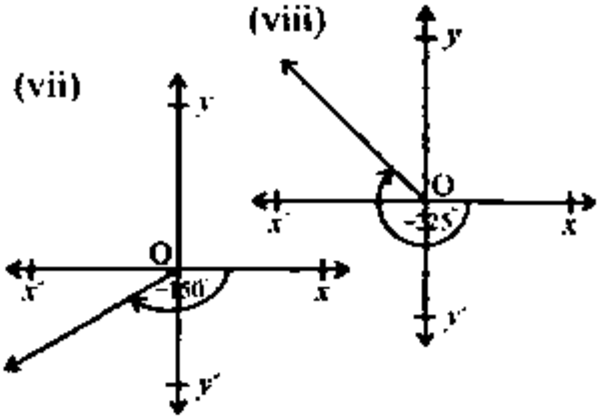


(v)



(vi)





Q.2 Express the following sexagesimal measures of angles in decimal form.

(i) $45^{\circ}30'$ (ii) $60^{\circ}\,30'30''$ (iii) $125^{\circ}\,22'50''$

Solution:

(i) $45^{\circ}30'$

$$45^{\circ} + \left(\frac{30}{60}\right)^{\circ}$$
$$45^{\circ} + .5^{\circ}$$
$$45.5^{\circ}$$

(ii) $60^{\circ}\,30'30''$

$$60^{\circ} + \left(\frac{30}{60}\right)^{\circ} + \left(\frac{30}{60 \times 60}\right)^{\circ}$$
$$60 + \left(\frac{1}{2}\right)^{\circ} + \left(\frac{1}{120}\right)^{\circ}$$
$$60^{\circ} + .5^{\circ} + .0083^{\circ}$$
$$60.5083^{\circ}$$

(iii) $125^{\circ}\,22'50''$

$$125^{o} + \left(\frac{22}{60}\right)^{o} + \left(\frac{50}{60\times 60}\right)^{o}$$

$$125^{o} + .37^{o} + .0139^{o}$$

$$125.3839^{o}$$

Q.3 Express the following into $D^{o}M'S'$ form.

(i) 47.36^{o} (ii) 125.45^{o} (iii) 225.75^{o}

(iv) -22.5^{o} (v) -67.58^{o} (vi) 315.18^{o}

Solution:

(i) 47.36^{o}

$$47^{o} + (.36)^{o}$$

$$47^{o} + \left(\frac{36}{100}\times 60\right)'$$

$$47^{o} + (21.6)'$$

$$47^{o} + 21' + .6'$$

$$47^{o} + 21' + \left(\frac{6}{10}\times 60\right)''$$

$$47^{o} + 21' + 36''$$

$$47^{o}\ 21'\ 36''$$

(ii) 125.45^{o}

$$125^{o} + (.45)^{o}$$

$$125^{o} +$$

$$125^{o} + \left(\frac{45}{100}\times 60\right)'$$

$$125^{o} + 27' = 125^{o}\ 27'$$

(iii) 225.75^{o}

$$225^{o} + (.75)^{o}$$

$$225^{o} + \left(\frac{75}{100}\right)^{o}$$

$= 225^{\circ} + \left(\frac{75}{100}\times 60\right)'$

$= 225^{\circ} \times \left(\frac{3}{4}\times 60\right)'$

$= 225^{\circ} + 45'$

$= 225^{\circ}\ 45'$

(iv) -22.5°

$-22^{\circ} - .5^{\circ}$

$= -22^{\circ} - \left(\frac{5}{10}\times 60\right)'$

$-22^{\circ} - 30'$

$-22^{\circ}\ 30'$

(v) -67.58°

$-67^{\circ} - (.58)^{\circ}$

$= -67^{\circ} - \left(\frac{58}{100}\times 60\right)'$

$= -67^{\circ} - 34.8'$

$= -67^{\circ} - 34' - .8'$

$= -67^{\circ} - 34' - \left(\frac{8}{10}\times 60\right)'$

$= -67^{\circ} - 34' - (48)''$

$= -67^{\circ}\ 34'\ (48)''$

(vi) 315.18°

$315^{\circ} + .18^{\circ}$

$= 315^{\circ} + \left(\frac{18}{100}\times 60\right)'$

$= 315^{\circ} + (10.8)'$

$= 315^{\circ} + 10' + \left(\frac{8}{10}\times 60\right)''$

$= 315^{\circ} + 10' + 48''$

$315^{\circ}\ 10'\ 48''$

Q.4 Express the following angles into radians

(i) 30^O (ii) 60^O (iii) 135^O
(iv) 225^O (v) -150^O (vi) -225^O
(vii) 300^O (viii) 315^O

Solution: **Note: $180^O = \pi$ radian**

(i) 30^O $\quad$ **$1^O = \frac{\pi}{180}$ radian**

$$30\left(\frac{\pi}{180}\right)$$

$$\frac{\pi}{6} \text{ radians}$$

$$\frac{22}{7 \times 6}$$

0.5238 radians

(ii) 60^O

$$60\left(\frac{\pi}{180}\right)$$

$$\frac{\pi}{3} \text{ radians}$$

$$\frac{22}{7 \times 3}$$

$$\frac{22}{21}$$

1.0476 radiAns.

(iii) 135^O

$$135^O = \overset{27}{135} \times \frac{\pi}{\underset{\underset{4}{36}}{180}}$$

$$\frac{3\pi}{4} \text{ radians}$$

$$\frac{3}{4} \times \frac{22}{7} \quad \frac{66}{28}$$

2.3571 radiAns.

(iv) 225^0

$$225 \times \frac{\pi}{180}$$

$$\frac{5\pi}{4} \text{ radians}$$

$$= \frac{5}{4} \times \frac{22}{7}$$

$$\frac{55}{14} \quad 3.92857 \text{ radians}$$

(v) -150^0

$$= -150 \times \frac{\pi}{180}$$

$$-\frac{5\pi}{6} \text{ radians}$$

$$= -\frac{5}{6} \times \frac{22}{7}$$

$$-\frac{55}{21} \quad -2.6190 \text{ radiAns.}$$

(vi) -225^0

$$-225 \times \frac{\pi}{180}$$

$= -225 \times \frac{\pi}{180}$

$-\frac{5\pi}{4}$ radians

$= -\frac{5}{4} \times \frac{22}{7}$

$-\frac{55}{14}$

-3.92857 radiAns.

(vii) 300^0

$= 300 \times \frac{\pi}{180}$

$\frac{5\pi}{3}$ radians

$\frac{5}{3} \times \frac{22}{7}$

$\frac{110}{21}$

5.2381 radiAns.

(viii) 315^0

$= 315 \times \frac{\pi}{180}$

$\frac{7\pi}{4}$ radians

$= \frac{7}{4} \times \frac{22}{7}$

$\frac{11}{2}$
5.5 radiAns.

Q.5 Convert each of the following radians to degrees

(i) $\frac{3\pi}{4}$ **(ii)** $\frac{5\pi}{6}$ **(iii)** $\frac{7\pi}{8}$

(iv) $\frac{13\pi}{16}$ **(v)** 3 **(vi)** 4.5

(vii) $\frac{-7\pi}{8}$ **(viii)** $-\frac{13}{16}\pi$

Solution:

π radians $= 180^{O}$

1 radian $= \left(\frac{180}{\pi}\right)^{O}$

(i) $\frac{3\pi}{4}$ **radians**

$\frac{3\pi}{4}\left(\frac{180}{\pi}\right)$
$\frac{3\times 180}{4}$
135^{O}

(ii) $\frac{5\pi}{6}$

$\frac{5\pi}{6}\times\frac{180}{\pi}$
$\frac{5\times 180}{6}$
150^{O}

(iii) $\frac{7\pi}{8}$

$\frac{7\pi}{8}\times\frac{180}{\pi}$
$\frac{7\times 180}{8}$
157.5^{O}

(iv) $\frac{13\pi}{16}$

$\frac{13\pi}{16} \times \frac{180}{\pi}$

$\frac{13}{16} \times 180$

146.25^{0}

(v) 3 radians

$3 \times \frac{180}{\pi}$

$= 3 \times \frac{180}{22} \times 7$

$\frac{1890}{11}$

171.818^{0}

(vi) 4.5 radians

$4.5 \times \frac{180}{\pi}$

$= \frac{45}{10} \times \frac{180}{22} \times 7$

$\frac{9 \times 45 \times 7}{11}$

$\frac{2835}{11}$

257.7273^{0}

(vii) $\frac{7\pi}{8}$

$$= \frac{-7\pi}{8} \times \frac{180}{\pi}$$

$$= \frac{315}{2}$$

$$-157.5^{O}$$

(viii) $\frac{13}{16}\pi$

$$= \frac{13}{16}\pi \times \frac{180}{\pi}$$

$$= -\frac{13 \times 180}{16}$$

$$-\frac{585}{4}$$

$$-146.25^{O}$$

EXERCISE 7.2

Formulae: (i) $\theta = \frac{l}{r}$

$$l = \theta r$$

Q.1 Find θ, when

(i) $l = 2cm$, $r = 3.5cm$

Solution:

$$\theta = \frac{l}{r}$$

$$\frac{2}{3.5}$$

$$\frac{2\times 10}{35}$$

0.5714 radiAns.

(ii) $l = 4.5$m, $r = 2.5$m

Solution:

$$\theta = \frac{l}{r}$$

$$= \frac{4.5}{2.5}$$

$$= \frac{45}{25}$$

1.8 radiAns.

Q.2 Find l, when

(i) $\theta = 180^\circ$, $r = 4.9$cm

Solution:

$$\theta = 180^\circ \times \frac{\pi}{180}$$

$= \pi$ radians

$$l = \theta r$$

$$= (\pi)(4.9)$$

$$= \frac{22}{7}\times\frac{49}{10}$$

$$= \frac{154}{10}$$

$= 15.4$ cm

(ii) $\theta = 60^\circ 30'$, $r = 15$mm

Solution:

$$\theta = 60^\circ 30'$$

$60^\circ + 30'$

$60^\circ + 30' \times \frac{1}{60}$

$60^\circ + .5^\circ$

60.5°

$60.5 \times \frac{\pi}{180}$ radian

$= \frac{\overset{121}{605}}{\underset{2}{10}} \times \frac{\pi}{180}$

$= \frac{121\pi}{2 \times 180}$

$l = \theta r$

$\left(\frac{121\pi}{2 \times 180}\right)(15)$

$= \frac{121 \times \overset{11}{22} \times \overset{1}{15}}{\underset{1}{2} \times \underset{12}{180} \times 7}$

$\frac{121 \times 11}{12 \times 7}$

$\frac{1331}{84}$

15.845 mm

Q.3 Find r, when

(i) $l = 4\text{cm}, \theta = \frac{1}{4}$ radians

$\theta = \frac{l}{r}$

$\therefore r = \frac{l}{\theta}$

$$\frac{4}{\frac{1}{4}}$$

$4 \times 4 = 16$ cm

(ii) $l = 52$cm, $\theta = 45^\circ$

Solution:

$$\theta = 45 \times \frac{\pi}{180}$$

$$= \frac{\pi}{4}$$

$$r = \frac{l}{\theta}$$

$$= \frac{52}{\frac{\pi}{4}}$$

$$= \frac{52 \times 4}{\frac{22}{7}}$$

$$= \frac{52 \times 4 \times 7}{22}$$

$$= \frac{728}{11}$$

$= 66.1818$ cm

Q.4 ***In a circle of radius 12 m, find the length of an arc which subtends a central angle $\theta = 1.5$ radian.***

Solution:

$r = 12$ m

$\theta = 1.5$ radian

$\because$ l ?

θ $\frac{l}{r}$

$\therefore$ l θr (putting values of θ, r)

(1.5) (12)

18 m

Q.5 In a circle of radius 10m, find the distance travelled by a point moving on this circle if the point makes 3.5 revolution.

Solution:

r 10 m

θ 3.5 revolutions

1 revolution - 2π radians

3.5 revolutions - $2\pi \times 3.5$

7π radians

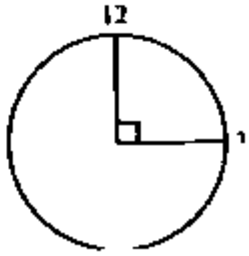
Distance travelled - l - θr

- $7\pi \times 10$

$7 \times \frac{22}{7} \times 10$ 220 m

Q.6 What is the circular measure of the angle between the hands of the watch at 3 O'clock?

Solution:



Measure of angle = 90^{O}

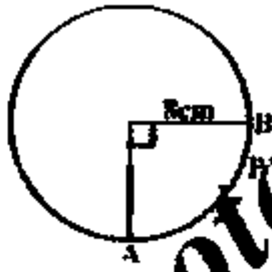
90^{O} - 90 ×

$\frac{\pi}{180}$ = $\frac{\pi}{2}$

radians

Q.7 What is the length of the arc APB.

Solution:



$\theta \quad 90^\circ$

$\quad 90 \times \frac{\pi}{180}$

$\quad \frac{\pi}{2}$ radians

$r \quad 8$ cm

$l \quad ?$

$l \quad \theta r$

$\quad \frac{\pi}{2} \times 8$

$\quad 4\pi$ cm

$\quad 4 \times \frac{22}{7}$

$\quad 12.571$ cm

Q.8 In a circle of radius 12cm, how long an arc subtends a central angle of 84^0.

Solution:

$r \quad 12$ cm

$\theta \quad 84^\circ$

$\quad = 84 \times \frac{\pi}{180}$

$\quad \frac{7\pi}{15}$ radians

$l \quad ?$

$l \quad \theta r$

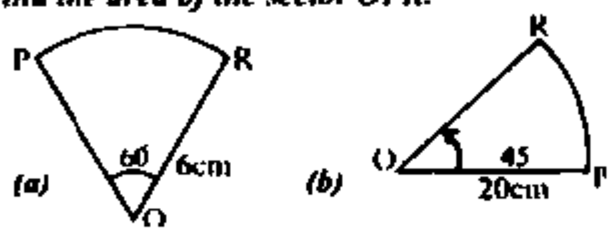
$\quad \frac{7\pi}{15} \times 12$

$\frac{7}{15} \times \frac{22}{7} \times 12$

$\frac{88}{5}$

17.6 cm

Q.9 Find the area of the sector OPR.



(a) *(b)*

Solution:

(a) Area of the sector $\frac{1}{2}r^2\theta$

Here, r 6 cm

θ 60°

$60 \times \frac{\pi}{180}$

θ $\frac{\pi}{3}$ radians

Area of the sector $\frac{1}{2}r^2\theta$

$\frac{1}{2}(6)^2(\frac{\pi}{3})$

$\frac{1}{2} \times 36 \times \frac{1}{3} \times \frac{22}{7}$

$6 \times \frac{22}{7}$

$\frac{132}{7}$

$= 18.857$ sq. cm.

(b) Area of the sector $= \frac{1}{2}r^2\theta$

Here, $r = 20$ cm

$\theta = 45^\circ$

$= 45 \times \frac{\pi}{180}$

$\theta = \frac{\pi}{4}$ radians

Area of the sector $= \frac{1}{2}r^2\theta$

$= \frac{1}{2}(20)^2(\frac{\pi}{4})$

$= \frac{1}{2} \times 400 \times \frac{\pi}{4}$

$= 50\pi$

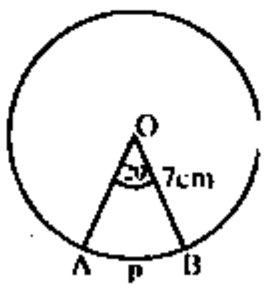
$= 50 \times \frac{22}{7}$

$= \frac{1100}{7}$

$= 157.14cm^2$

Q.10 Find area of the sector inside a central angle of 20° in a circle of radius 7 m.

Solution:



(a) Area of the sector $= \frac{1}{2}r^2\theta$

Here, $r = 7$ cm

$\theta = 20^\circ$

$= 20 \times \frac{\pi}{180}$

θ $\frac{\pi}{9}$ radians

Area of the sector $\frac{1}{2}r^2\theta$

$\frac{1}{2}(7)^2(\frac{\pi}{9})$

$\frac{1}{2}\times 49\times\frac{\pi}{9}$

$\frac{49}{18}\pi$

$= \frac{49}{18}\times\frac{22}{7}$

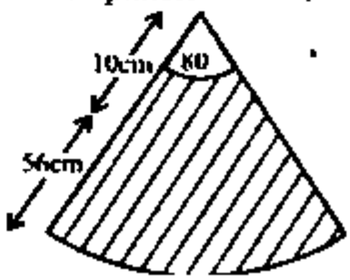
$\frac{77}{9}$

8.555

8.6 sq. cm. (approx)

Q.11 Sehar is making a skirt. Each panel of this skirt is of the shape shown shaded in the diagram. How much material (cloth) is required for each panel?

Solution:



(a) Area of the sector $\frac{1}{2}r^2\theta$

Here, r 56 + 10

66 cm

θ 80°

80 ×

$\frac{\pi}{180}$

θ $\frac{4\pi}{9}$ radians

Area of the whole sector $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} (66)^2 (\frac{4\pi}{9})$

$= \frac{1}{2} \times 66 \times 66 \times \frac{4\pi}{9}$

$= 11 \times 22 \times 4\pi$

$= 968\pi$ sq. cm.

Area of the upper sector $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} (10)^2 (\frac{4\pi}{9})$

$= \frac{1}{2} (10)(10)(\frac{4\pi}{9})$

$= \frac{200\pi}{9}$

Area of the panel $= 968\pi - \frac{200\pi}{9}$

$= \frac{8712\pi - 200\pi}{9}$

$= \frac{8512\pi}{9}$

$= \frac{8512}{9} \times \frac{22}{7}$

$= 2972.44$

$= 2972$ sq. cm.

Q.12 Find the area of the sector with central angle of $\frac{\pi}{5}$ radian in a circle of radius 10cm.

Solution:

(a) Area of the sector $\frac{1}{2}r^2\theta$

Here, $r = 10$ cm

$\theta = \frac{\pi}{5}$ radians

Area of the sector $= \frac{1}{2}r^2\theta$

$= \frac{1}{2}(10)^2(\frac{\pi}{5})$

$= \frac{1}{2}\times 10\times 10\times\frac{\pi}{5}$

$= 10\pi$ sq. cm.

$= 10\times\frac{22}{7}$

$= 31.4286$ sq. cm.

Q.13 The area of the sector with central angle θ in a circle of radius 2m is 10 sq meter. Find θ in radiAns.

Solution:

Area of the sector $= \frac{1}{2}r^2\theta$ (θ is in radians)

Here, $A = 10$ sq. m

$r = 2$ m

$A = \frac{1}{2}r^2\theta$

Putting values of A and r.

$10 = \frac{1}{2}(2)^2(\theta)$

$$10 = \frac{4}{2}(\theta)$$

$$10 = 2\theta$$

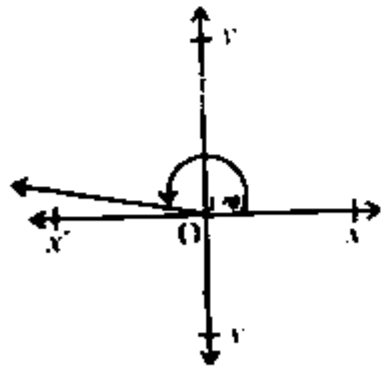
$$\theta = 5 \text{ radians}$$

EXERCISE 7.3

1. ***Locate each of the angles in standard position using a protractor or fair free hand guess. Also find a positive and a negative angle coterminal with each given angle.***

(i) 170^{0}

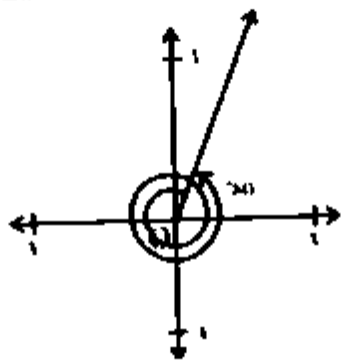
Sol.



Positive Coterminal angle is
$360^{0} + 170^{0} = 530^{0}$
Negative Coterminal angle is
-190^{0}

(ii) 780^{0}

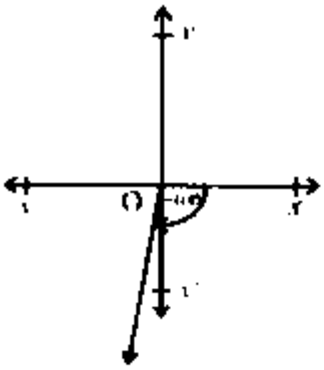
Sol.



$780^{0} = 360^{0} + 360^{0} + 60^{0}$
Positive Coterminal angle
60^{0}
Negative Coterminal angle is
-300^{0}

(iii) 100^{O}

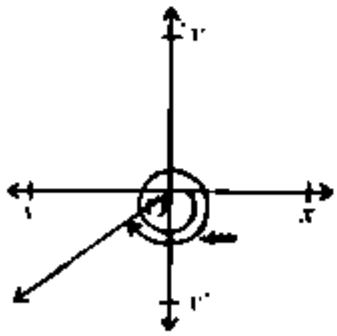
Sol.



Positive Coterminal angle is
$360 - 100 = 260^{O}$
Negative Coterminal angle is
$-360 - 100 = -460^{O}$

(iv) -500^{O}

Sol.



Positive Coterminal angle
$360 - 140 = 220^{O}$
Negative Coterminal angle is
-140
$(-360 - 140 = -500)$

2. ***Identify the closest quadrantal angles between which the following angles lies.***

(i) 156^{O} (ii) 318^{O} (iii) 572^{O} (iv) -330^{O}

Solution:

(i) 156^{O}

$\boxed{90^{O}, 180^{O}}$

(ii) 318^{O}

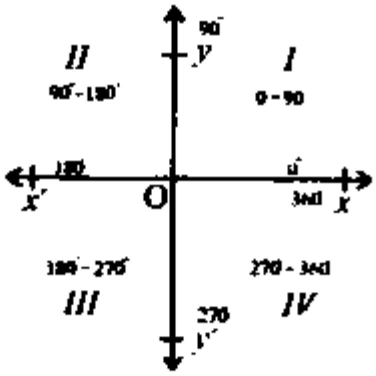
$\boxed{270^{O}, 360^{O}}$

(iii) 572^{O}

$\boxed{540^{O}, 630^{O}}$

(iv) -330^{O}

$\boxed{0^{O}, 90^{O}}$



3. ***Write the closest quadrantal angles between which the angle lies. Write your answer in radian measure.***

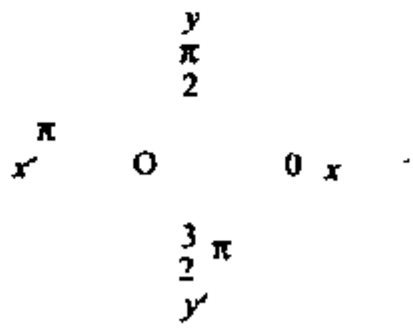
(i) $\frac{\pi}{3}$ (ii) $\frac{3\pi}{4}$ (iii) $\frac{-\pi}{4}$ (iv) $\frac{-3\pi}{4}$

Solution:

(i) $\frac{\pi}{3}$

$0 < \frac{\pi}{3} < \frac{\pi}{2}$

quadrant: $\boxed{0, \frac{\pi}{2}}$



(ii) $\frac{3\pi}{4}$

$\frac{\pi}{2} < \frac{3\pi}{4} < \pi$

quadrant: $\boxed{\frac{\pi}{2}, \pi}$

(iii) $-\frac{\pi}{4}$

$-\frac{\pi}{2} < -\frac{\pi}{4} < 0$

$\boxed{0, \frac{-\pi}{2}}$

(iv) $-\frac{3\pi}{4}$

$-\pi < -\frac{3\pi}{4} < -\frac{\pi}{2}$

$\boxed{-\frac{\pi}{2}, -\pi}$

4. ***In which quadrant θ lie when***

(i) $\sin\theta > 0$, $\tan\theta < 0$ *(ii)* $\cos\theta < 0$, $\sin\theta < 0$

(iii) $\sec\theta > 0$, $\sin\theta < 0$ *(iv)* $\cos\theta < 0$, $\tan\theta < 0$

(v) $\csc\theta > 0, \cos\theta > 0$ *(vi)* $\sin\theta < 0$, $\sec\theta < 0$

Solution:

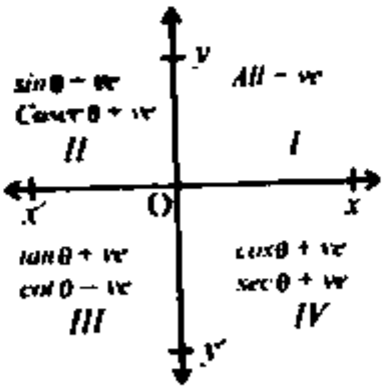
(i) $\sin\theta > 0$

$\tan\theta < 0$

Quadrant: $\boxed{\text{II}}$

as $\sin\theta$ is + ve

$\tan\theta$ is − ve in II



(ii) $\cos\theta < 0$

$\sin\theta < 0$

Quadrant: $\boxed{\text{III}}$

$\cos\theta$ is − ve

$\sin\theta$ is − ve in III

(iii) $\sec\theta > 0$

$\sin\theta < 0$

Quadrant: $\boxed{\text{IV}}$

$\sec\theta$ is + ve

(iv) $\cos\theta < 0$

$\tan\theta < 0$

Quadrant: $\boxed{\text{II}}$

$\cos\theta$ is − ve

	$\sin\theta$ is $-$ ve in IV		$\tan\theta$ is $-$ ve in II
(v)	$\text{cosec}\,\theta > 0$	*(vi)*	$\sin\theta < 0$
	$\cos\theta > 0$		$\sec\theta < 0$
	Quadrant: I		Quadrant: $\boxed{\text{III}}$
	$\text{cosec}\,\theta$ + ve		$\sin\theta$ is $-$ ve
	$\cos\theta$ + ve in I		$\sec\theta$ is $-$ ve in III

5. ***Fill in the blanks.***

(i) $\cos(-150^{\circ})$ = ... $\cos 150^{\circ}$

(ii) $\sin(-310^{\circ})$ = ... $\sin 310^{\circ}$

(iii) $\tan(-210^{\circ})$ = ... $\tan 210^{\circ}$

(iv) $\cot(-45^{\circ})$ = ... $\cot 45^{\circ}$

(v) $\sec(-60^{\circ})$ .... $\sec 60^{\circ}$

(vi) $\text{cosec}(-137^{\circ})$ = ... $\text{cosec}\,137^{\circ}$

Solution:

(i) $\cos(-150^{\circ}) = + \cos 150^{\circ}$

(ii) $\sin(-310^{\circ}) = -\sin 310^{\circ}$

(iii) $\tan(-210^{\circ}) = -\tan 210^{\circ}$

(iv) $\cot(-45^{\circ}) = -\cot 45^{\circ}$

(v) $\sec(-60^{\circ}) = +\sec 60^{\circ}$

(vi) $\text{cosec}(-137^{\circ}) = -\text{cosec}\,137^{\circ}$

6. ***The given point P lies on the terminal side of θ. Find quadrant of θ and all six trigonometric ratios.***

(i) $(-2,3)$ *(ii)* $(-3,-4)$ *(iii)* $(\sqrt{2}, 1)$

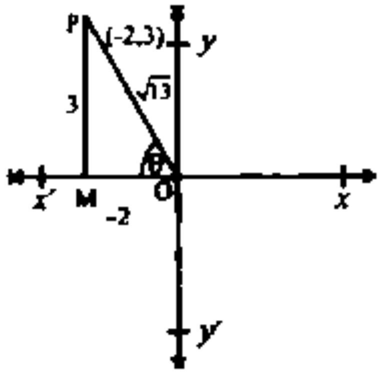
Solution:

From Δ PMO (Pythagorean Theorem)

in $\overline{PO} = \sqrt{(-2)^2+(3)^2}$

$= \sqrt{4+9}$

$= \sqrt{13}$



θ lies in II quadrant.

$\sin\theta = \frac{3}{\sqrt{13}}$,

$\text{cosec}\theta = \frac{\sqrt{13}}{3}$, $\cos\theta = \frac{-2}{\sqrt{13}}$, $\sec\theta = -\frac{\sqrt{13}}{2}$,

$\tan\theta = \frac{-3}{2}$, $\cot\theta = -\frac{2}{3}$

(ii) **P(−3, −4)**

From Δ PMO

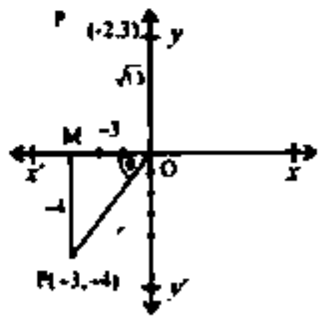
Pythagorean Theorem

$r = \sqrt{(-3)^2+(-4)^2}$

$= \sqrt{9+16}$

$= \sqrt{25}$

5



θ lies in III quadrant.

$\sin\theta = -\frac{4}{5}$, $\text{cosec}\theta = -\frac{5}{4}$, $\cos\theta = -\frac{3}{5}$, $\sec\theta = -\frac{5}{3}$,

$\tan\theta = \frac{4}{3}$, $\cot\theta = \frac{3}{4}$

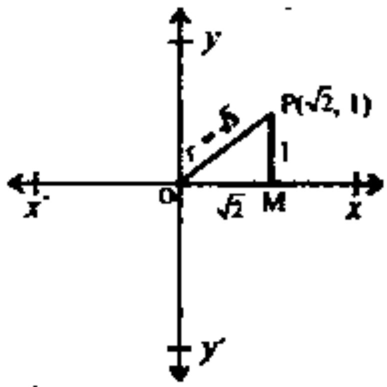
(iii) $P(\sqrt{2}, 1)$

From Δ PMO

Pythagorean Theorem

$$r = \sqrt{(\sqrt{2})^2 + (1)^2}$$
$$= \sqrt{2+1}$$
$$= \sqrt{3}$$

θ lies in I quadrant.



$$\sin\theta = \frac{1}{\sqrt{3}}, \text{cosec}\theta = \sqrt{3}, \cos\theta = \sqrt{\frac{2}{3}}, \sec\theta = \sqrt{\frac{3}{2}},$$
$$\tan\theta = \frac{1}{\sqrt{2}}, \cot\theta = \sqrt{2}$$

7. ***If $\cos\theta = \frac{-2}{3}$ and terminal arm of the angle θ is in quadrant II, find the values of remaining trigonometric functions.***

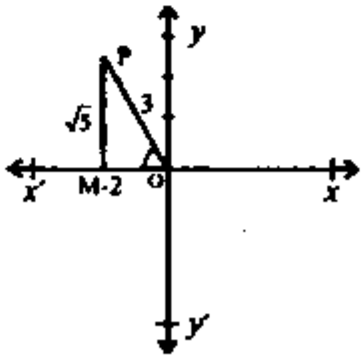
In Δ PMO

in $\overline{PO} = 3$

in $\overline{MO} = 2$

Now, By Pythagorean Theorem

$$\text{in } \overline{PM} = \sqrt{(m\,\overline{PO})^2 - (m\,\overline{MO})^2}$$
$$= \sqrt{(3)^2 - (2)^2}$$
$$= \sqrt{9-4}$$



$$= \sqrt{5}$$

$$\sin\theta = \frac{\sqrt{5}}{3}, \text{cosec}\theta = \frac{3}{\sqrt{5}}, \tan\theta = \frac{-\sqrt{5}}{2}, \cot\theta = \frac{-2}{\sqrt{5}},$$

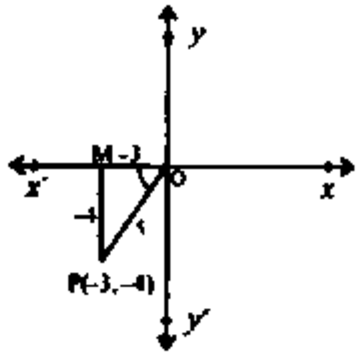
$$\sec\theta = -\frac{3}{2}$$

8. ***If $\tan\theta = \frac{4}{3}$ and $\sin\theta < 0$, find the values of other trigonometric functions at θ.***

Solution: $\tan\theta$ is + ve in third quadrant where as $\sin\theta$ is negative i.e., < 0; therefore, θ lies in III quadrant.

From Δ OMP

$$r = \sqrt{(3)^2 + (4)^2}$$
$$= \sqrt{9+16}$$
$$= \sqrt{25}$$
$$= 5$$



$$\sin\theta = \frac{-4}{5}.$$

$$\csc\theta = -\frac{5}{4}, \cos\theta = -\frac{3}{5}, \sec\theta = -\frac{5}{3}, \cot\theta = \frac{3}{4}$$

9. ***If $\sin\theta = \frac{-1}{\sqrt{2}}$ and terminal side of the angle is not in quadrant III, find the values of $\tan\theta$, $\sec\theta$, and $\csc\theta$.***

Solution: $\sin\theta$ is negative in III and IV quadrant, therefore, θ lies in IV quadrant as it does not lie in III quadrant.

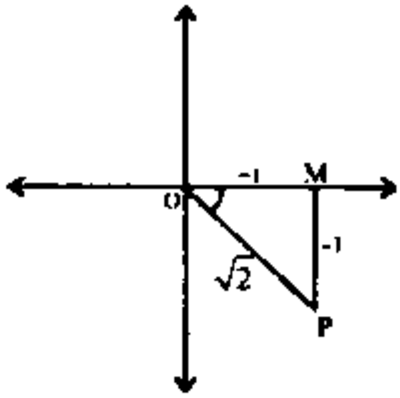
$\sin\theta = \frac{1}{\sqrt{2}}$

From Δ OMP

$m\overline{OM} = \sqrt{\left(\sqrt{2}\right)^2 - (1)^2}$

$= \sqrt{2-1}$

$= 1$



Now, $\tan\theta = \frac{-1}{1} = -1$

$\sec\theta = \frac{\sqrt{2}}{1} = \sqrt{2}$

$\text{cosec}\,\theta = -\sqrt{2}$

10. If $\text{cosec}\,\theta = \frac{13}{12}$ and $\sec\theta > 0$, find the remaining trigonometric functions.

Solution: Cosecθ is + ve and secθ is + ve in 1st quadrant, therefore, θ lies in first quadrant.

$\text{cosec}\,\theta = \frac{13}{12}$

From OMP triangle.

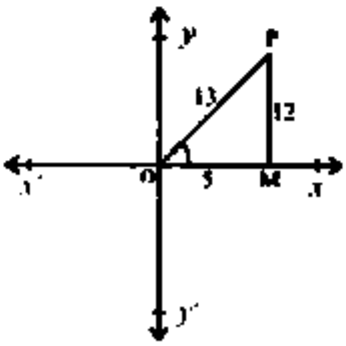
$mOM = \sqrt{(mOP)^2 - (mMP)^2}$

$= \sqrt{(13)^2 - (12)^2}$

$= \sqrt{169 - 144}$

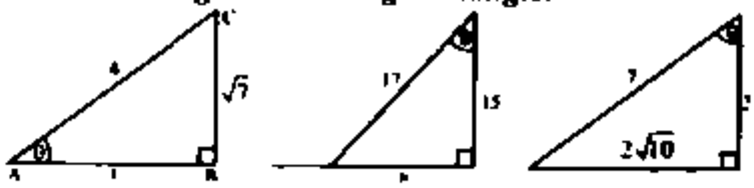
$= \sqrt{25}$

$= 5$



$$\sin\theta = \frac{12}{13}, \cos\theta = \frac{5}{13}, \sec\theta = \frac{13}{5}, \tan\theta = \frac{12}{5}, \cot\theta = \frac{5}{12}$$

11. Find the values of trigonometric functions at the indicated angle θ in the right triangle.



(i) *(ii)* *(iii)*

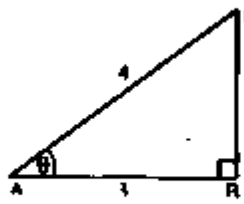
Solution: Fig (i)

$$(m\overline{BC})^2 = (m\overline{AC})^2 - (m\overline{AB})^2$$

$$= \sqrt{(4)^2 - (3)^2}$$

$$= \sqrt{16-9}$$

$$= \sqrt{7}$$



$$\sin\theta = \frac{\sqrt{7}}{4}, \operatorname{cosec}\theta = \frac{4}{\sqrt{7}}, \cos\theta = \frac{3}{4}, \sec\theta = \frac{4}{3},$$

$$\tan\theta = \frac{\sqrt{7}}{3}, \cot\theta = \frac{3}{\sqrt{7}}$$

(ii) **From fig (ii)**

$$\sin\theta = \frac{8}{17} \qquad \operatorname{cosec}\theta = \frac{17}{8}$$

$$\cos\theta = \frac{15}{17} \qquad \sec\theta = \frac{17}{15}$$

$$\tan\theta = \frac{8}{15} \qquad \cot\theta = \frac{15}{8}$$

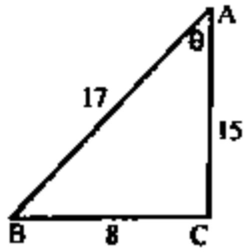


Fig (ii)

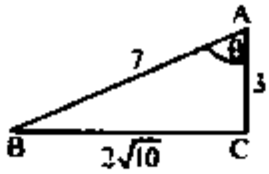
(iii) **From fig (iii)**

$$(m\overline{BC})^2 = (m\overline{AB})^2 - (m\overline{AC})^2$$

$$m\overline{BC} = \sqrt{49-9}$$

$$= \sqrt{40}$$

$$= 2\sqrt{10}$$



$\sin\theta \quad \frac{2\sqrt{10}}{7}$ $\quad \operatorname{cosec}\theta = \frac{7}{2\sqrt{10}}$

$\cos\theta \quad \frac{3}{7}$ $\quad \sec\theta = \frac{7}{3}$

$\tan\theta \quad \frac{2\sqrt{10}}{3}$ $\quad \cot\theta = \frac{3}{2\sqrt{10}}$

12. ***Find the values of the trigonometric functions. Do not use trigonometric tables or calculator.***

(i) **$\tan 30^{O}$**

Ans. We know that $2k\pi + \theta = \theta$ where $k \in Z$

$$\tan 30^{O} \quad \tan\left(2\pi + \frac{\pi}{6}\right) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

(ii) **$\tan 330^{O}$**

Ans. $\tan(360^{O} - 30^{O})$

$$= \tan\left(2\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$$

(iii) **$\sec 330^{O}$**

Ans. $\sec(360 - 30)$

$$= \sec\left(2\pi - \frac{\pi}{6}\right) = \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$$

(iv) $\cot\frac{\pi}{4}$

Ans. $=\cot\left(2\pi+\frac{\pi}{4}\right)=\cot\frac{\pi}{4}=1$

(v) $\cos\frac{2\pi}{3}$

Ans. $=\cos\left(\pi-\frac{\pi}{3}\right)=-\cos\frac{\pi}{3}=-\frac{1}{2}$

(vi) $\operatorname{cosec}\frac{2\pi}{3}$

Ans. $=\operatorname{cosec}\left(\pi-\frac{\pi}{3}\right)=\operatorname{cosec}\frac{\pi}{3}=\frac{2}{\sqrt{3}}$

(vii) $\cos(-450^{o})$

Ans. $=\cos(-360-90^{o})$

$=\cos\left[-2\pi-\frac{\pi}{2}\right]$

$=\cos\left(-\frac{\pi}{2}\right)=0$

(viii) $\tan(-9\pi)$

Ans. $=\tan(-9\pi)$

$=\tan(-8\pi-\pi)$

$=\tan(-\pi)=\infty$

(ix) $\cos\left(\frac{-5\pi}{6}\right)$

Ans. $=\cos\left(-\pi+\frac{\pi}{6}\right)$

$= -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$

(x) $\sin\frac{7\pi}{6}$

Ans. $= \sin\left(\pi + \frac{\pi}{6}\right)$

$= -\sin\frac{\pi}{6}$

$= -\frac{1}{2}$

(xi) $\cot\frac{7\pi}{6}$

$= \cot\left(\pi + \frac{\pi}{6}\right)$

$= \cot\frac{\pi}{6}$

$\sqrt{3}$

(xii) $\cos 225^{\circ}$

$= \cos\left(\frac{5\pi}{4}\right)$

$= \cos\left(\pi + \frac{5\pi}{4}\right)$

$= -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

Important Relations

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \mathrm{cosec}^2\theta$

EXERCISE 7.4

In Problems 1 – 6, simplify each expression to a single trigonometric function.

1. $\dfrac{\sin^2\theta}{\cos^2\theta}$

Sol. $\dfrac{\sin\theta}{\cos\theta} \times \dfrac{\sin\theta}{\cos\theta}$

$\tan\theta \times \tan\theta$

$\tan^2\theta$

2. $\tan x \sin x \sec x$

Sol. $= \dfrac{\sin x}{\cos x} \times \sin x \times \dfrac{1}{\cos x}$

$= \dfrac{\sin x}{\cos x} \times \dfrac{\sin x}{\cos x}$

$= \tan x \times \tan x$

$= \tan^2 x$

3. $\dfrac{\tan x}{\sec x}$

Sol. $= \tan x \times \cos x$ $\left[\cos x = \dfrac{1}{\sec x}\right]$

$= \dfrac{\sin x}{\cos x} \times \cos x$

$\sin x$

4. $1 - \cos^2 x$

Sol. $= \sin^2 x + \cos^2 x - \cos^2 x$ $\left[\sin^2\theta + \cos^2\theta = 1\right]$

$\sin^2 x$

5. $\sec^2 x - 1$

Sol. $= 1 + \tan^2 x - 1$

$= \tan^2 x$

6. $\sin^2 x \cdot \cot^2 x.$

Sol. $= \sin^2 x \times \frac{\cos^2 x}{\sin^2 x}$

$= \cos^2 x$

In problems 7 – 24, verify the identities.

7. $(1 - \sin\theta)(1 + \sin\theta) = \cos^2\theta$

Sol. Takig L.H.S.

$(1 - \sin\theta)(1 + \sin\theta)$

$= (1)^2 - \sin^2\theta$

$= 1 - \sin^2\theta$

$= \cos 2 = \text{R. H. S.}$ $\left[\text{using } \sin^2\theta + \cos^2\theta = 1\right]$

8. $\frac{\sin\theta + \cos\theta}{\cos\theta} = 1 + \tan\theta$

Sol. Taking L.H.S.

$\frac{\sin\theta + \cos\theta}{\cos\theta}$

$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\cos\theta}$

$= \tan\theta + 1$

$= 1 + \tan\theta = R.H.S.$

9. $(\tan\theta + \cot\theta)\tan\theta = \sec^2\theta$

Sol. Taking L.H.S.

$(\tan\theta + \cot\theta)(\tan\theta)$

$= \tan\theta\tan\theta + \cot\theta\ \tan\theta$

$= \tan^2\theta + \cot\theta \times \frac{1}{\cot\theta}$

$\tan^2\theta + 1 = \sec^2\theta = $ R. H. S. [using $1 + \tan^2\theta = \sec^2\theta$]

10. $(\cot\theta + \text{cosec}\,\theta)(\tan\theta - \sin\theta) = \sec\theta - \cos\theta$

Sol. Taking L. H. S.

$(\cot\theta + \text{cosec}\,\theta)(\tan\theta - \sin\theta)$

$$= \left(\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}\right)\left(\frac{\sin\theta}{\cos\theta} - \sin\theta\right)$$

$$= \left(\frac{\cos\theta + 1}{\sin\theta}\right)\left(\frac{\sin\theta - \sin\theta\cos\theta}{\cos\theta}\right)$$

$$= \frac{\sin\theta\cos\theta - \sin\theta\cos^2\theta + \sin - \sin\theta\cos\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin\theta - \sin\theta\cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin\theta(1 - \cos^2\theta)}{\sin\theta\cos\theta}$$

$$= \frac{1 - \cos^2\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}$$

$= \sec\theta - \cos\theta = $ R.H.S.

11. $\dfrac{\sin\theta + \cos\theta}{\tan^2\theta - 1} = \dfrac{\cos^2\theta}{\sin\theta - \cos\theta}$

Sol. Taking L.H.S.

$$\frac{\sin\theta + \cos\theta}{\dfrac{\sin^2\theta}{\cos^2\theta} - 1}$$

$$= \frac{(\sin\theta + \cos\theta)\cos^2\theta}{\sin^2\theta - \cos^2\theta}$$

$$= \frac{(\sin\theta + \cos\theta)\cos^2\theta}{(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)}$$

$$= \frac{\cos^2\theta}{\sin\theta - \cos\theta} = R.H.S.$$

12. $\frac{\cos^2\theta}{\sin\theta} + \sin\theta = \operatorname{cosec}\theta$

Sol. Taking L.H.S.

$$= \frac{\cos^2\theta}{\sin\theta} + \sin\theta$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta} \quad \left[\text{using } \sin^2 + \cos^2\theta = 1\right]$$

$$= \frac{1}{\sin\theta}$$

$$\operatorname{cosec}\theta = R.H.S.$$

13. $\sec\theta - \cos\theta = \tan\theta \sin\theta$

Sol. Taking L.H.S.

$$\sec\theta - \cos\theta$$

$$= \frac{1}{\cos\theta} - \cos\theta$$

$$= \frac{1 - \cos^2\theta}{\cos\theta} \quad \left[\text{using } \sin^2\theta + \cos^2\theta = 1\right]$$

$$= \frac{\sin^2\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta} \times \sin\theta$$

$$= \tan\theta \sin\theta = \text{R.H.S.}$$

14. $\dfrac{\sin^2\theta}{\cos\theta} + \sin\theta = \sec\theta$

Sol. Taking L.H.S.

$$\frac{\sin^2\theta}{\cos\theta} + \cos\theta$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} \qquad \left[\sin^2\theta + \cos^2\theta = 1\right]$$

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta = \text{R.H.S.}$$

15. $\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta$

Sol. Taking L.H.S.

$$\tan\theta + \cot\theta$$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta \sin\theta}$$

$$= \frac{1}{\cos\theta \sin\theta}$$

$$= \sec\theta \ \operatorname{cosec}\theta = \text{R.H.S.}$$

16. $(\tan\theta + \cot\theta)(\cos\theta + \sin\theta) = \sec\theta + \operatorname{cosec}\theta$

Sol. Taking L.H.S.

$$(\tan\theta + \cot\theta)(\cos\theta + \sin\theta)$$

$$= \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)(\cos\theta + \sin\theta)$$

$$= \left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta \ \sin\theta}\right)(\cos\theta + \sin\theta)$$

$$= \left(\frac{1}{\cos\theta \sin\theta}\right)(\cos\theta + \sin\theta)$$

$$= \left(\frac{\cos\theta + \sin\theta}{\cos\theta \sin\theta}\right)$$

$$= \frac{\cos\theta}{\cos\theta \sin\theta} + \frac{\sin\theta}{\cos\theta \sin\theta}$$

$$= \frac{1}{\sin\theta} + \frac{1}{\cos\theta}$$

$$= \operatorname{cosec}\theta + \sec\theta$$

$$= \sec\theta + \operatorname{secec}\theta = \text{R.H.S.}$$

17. $\sin\theta(\tan\theta + \cot\theta) = \sec\theta$

Sol. Taking L.H.S.

$$\sin\theta(\tan\theta + \cot\theta)$$

$$= \sin\theta\left[\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right]$$

$$= \sin\theta\left[\frac{\sin^2\theta + \cos^2\theta}{\cos\theta \sin\theta}\right]$$

$$= \frac{(\sin\theta)(1)}{\cos\theta \sin\theta}$$

$$= \frac{1}{\cos\theta}$$

$$\sec\theta \quad \text{R.H.S.}$$

18. $\dfrac{1+\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{1+\cos\theta} = 2\operatorname{cosec}\theta$

Sol. Taking L.H.S.

$$\frac{1+\cos\theta}{\sin\theta} + \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{1+\cos^2\theta + 2\cos\theta + \sin^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{1+2\cos\theta + (\sin^2\theta + \cos^2\theta)}{\sin\theta(1+\cos\theta)}$$

$$= \frac{1+2\cos\theta + 1}{(\sin\theta)(1+\cos\theta)}$$

$$= \frac{2+2\cos\theta}{(\sin\theta)(1+\cos\theta)}$$

$$= \frac{2(1+\cos\theta)}{(\sin\theta)(1+\cos\theta)}$$

$$= \frac{2}{\sin\theta}$$

$2\ \mathrm{cosec}\theta$ = R.H.S.

19. $\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} = 2\mathrm{cosec}^2\theta$

Sol. Taking L.H.S.

$$\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}$$

$$= \frac{1+\cos\theta + 1 - \cos\theta}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{2}{1-\cos^2\theta}$$

$$= \frac{2}{\sin^2\theta} \quad \left[\text{using } \sin^2\theta + \cos^2\theta = 1\right]$$

$= 2\mathrm{cosec}^2\theta$ = R.H.S.

20. $\frac{1+\sin\theta}{1-\sin\theta} - \frac{1-\sin\theta}{1+\sin\theta} = 4\tan\theta\sec\theta$

Sol. Taking L.H.S.

$$\frac{1+\sin\theta}{1-\sin\theta}-\frac{1-\sin\theta}{1+\sin\theta}$$
$$=\frac{(1+\sin\theta)^2-(1-\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}$$
$$=\frac{(1+2\sin\theta+\sin^2\theta)-(1-2\sin\theta+\sin\theta^2)}{(1-\sin\theta)(1+\sin\theta)}$$
$$=\frac{1+2\sin\theta+\sin^2\theta-1+2\sin\theta-\sin^2\theta}{(1^2-\sin^2\theta)}$$
$$=\frac{4\sin\theta}{\cos^2\theta}\quad\left[\text{using } \sin^2\theta+\cos^2\theta=1\right]$$
$$=\frac{4\sin\theta}{\cos\theta\cos\theta}$$
$$=4\tan\theta\sec\theta=\text{R.H.S.}$$

21. $\sin^3\theta=\sin\theta-\sin\theta\cos^2\theta$

Sol. Taking R.H.S.

$$=\sin\theta-\sin\theta\cos^2\theta$$
$$=\sin\theta\left[1-\cos^2\theta\right]$$
$$=(\sin\theta)(\sin^2\theta)\left[\text{using } \sin^2\theta+\cos^2\theta=1\right]$$
$$=\sin^3\theta=\text{L.H.S.}$$

22. $\cos^4\theta-\sin^4\theta=(\cos^2\theta-\sin^2\theta)$

Sol. L.H.S.

$$=(\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)\quad \cos^2\theta+\sin^2\theta=1$$
$$=(1)\quad\cos^2\theta-\sin^2\theta$$
$$=(\cos^2\theta-\sin^2\theta)=\text{R.H.S.}\quad\text{Proved}$$

23. $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}=\frac{\sin\theta}{1-\cos\theta}$

Sol. Taking L.H.S.

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

$$\sqrt{\frac{(1+\cos\theta)}{(1-\cos\theta)}\times\frac{(1-\cos\theta)}{(1-\cos\theta)}} \quad N.T.S.$$

$$\sqrt{\frac{(1+\cos\theta)(1-\cos\theta)}{(1-\cos\theta)^2}}$$

$$=\frac{\sqrt{1-\cos^2\theta}}{1-\cos\theta}$$

$$\frac{\sqrt{\sin^2\theta}}{1-\cos\theta} \quad \left[\text{using } \sin^2\theta+\cos^2\theta=1\right]$$

$$\frac{\sin\theta}{1-\cos\theta}=\text{R.H.S.}$$

24. $\sqrt{\dfrac{\sec\theta+1}{\sec\theta-1}}=\dfrac{\sec\theta+1}{\tan\theta}$

Sol. Taking L.H.S.

$$\sqrt{\frac{\sec\theta+1}{\sec\theta-1}}$$

$$\sqrt{\frac{(\sec\theta+1)(\sec\theta+1)}{(\sec\theta-1)(\sec\theta+1)}} \quad \text{N.T.Step}$$

$$\sqrt{\frac{(\sec\theta+1)^2}{\sec^2\theta-1^2}}$$

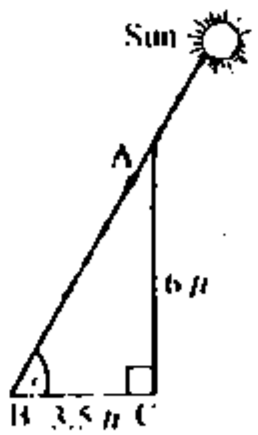
$$\frac{\sec\theta+1}{\sqrt{\tan^2\theta}} \quad \left[\text{using } 1+\tan^2\theta=\sec^2\theta\right]$$

$$=\frac{\sec\theta+1}{\tan\theta}=\text{R.H.S.}$$

EXERCISE 7.5

1. ***Find the angle of elevation of the sun if a 6 feet man casts a 3.5 feet shadow.***

Solution: Let AC be the man and BC be his shadow. From the figure



$$\tan\theta = \frac{m\overline{AC}}{m\overline{BC}}$$

$$= \frac{6}{3.5}$$

$$\tan\theta = 1.71429$$

$$\theta = \tan^{-1} 1.71429$$

(Using Calculator)

$$= 59.74^{0}$$

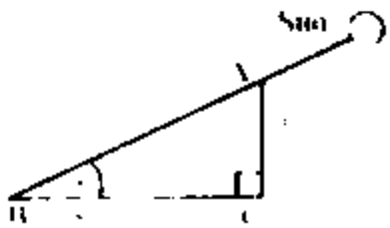
$$= 59^{0}\,44'\,24''$$

2. ***A tree casts a 40 meter shadow when the angle of elevation of the sun is 25^{0}. Find the height of the tree.***

Solution: Let AC be the tree and it height as h meter.

Shadow of the tree is $m\overline{BC} = 40m$.

Now, from the figure



$$\tan 25^{0} = \frac{h}{40}$$

$$0.4663 = \frac{h}{40}$$

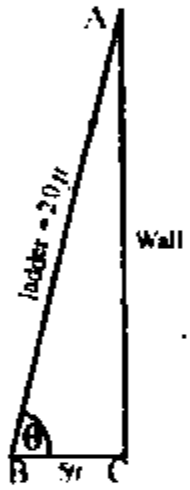
$$h = 40 \times 0.4663$$

(using calculator)

$$= 18.652\ m$$

3. ***A 20 feet long ladder is leaning against a wall. The bottom of the ladder is 5 feet from the base of the wall. Find the acute angle (angle of elevation) the ladder makes with the ground.***

Solution: Let $\overline{AC}$ be the wall and $\overline{AB} = 20$ ft be the ladder and $m\overline{BC} = 5$ ft.



then
$$\text{Cos}\theta = \frac{m\overline{BC}}{m\overline{AB}}$$
$$= \frac{5}{20}$$
$$\text{Cos}\theta = 0.25$$
$$\theta = \text{Cos}^{-1}.25$$
$$\theta = 75.52225^{\circ}$$
(using calculator)
$$= 75.5^{\circ} \text{ Approx}$$
$$= 75^{\circ}\ 30'$$

4. ***The base of a rectangle is 25 feet and the height of the rectangle is 13 feet. Find the angle that the diagonal of the rectangle makes with the base.***

Solution: Let ABCD be the rectangle in which $m\overline{AB} = 25$ ft, $m\overline{CB} = 13$ ft and α be the angle that diagonal $\overline{AC}$ makes with $\overline{AB}$

$\tan\alpha = \frac{m\overline{CB}}{m\overline{AB}}$

$= \frac{13}{25}$

$\tan\alpha = 0.52$

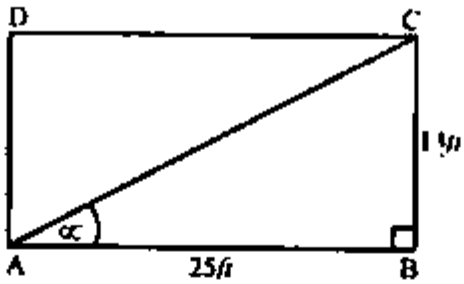
$\alpha = \tan^{-1}0.52$

$= 27.4744^{O}$

(using calculator)

27.47^{O}

$27^{O}\,28'$



5. ***A rocket is launched and climbs at a constant angle of 80^{O}. Find the altitude of the rocket after it travels 5000 meter.***

Solution: Suppose $\overline{BA} = 5000$ m be the distance covered by the rocket. 80^{O} angle is made by the rocket with the base $\overline{BC}$. $\overline{AC}$ is altitude.

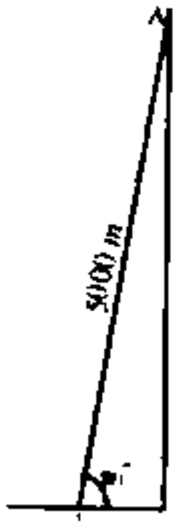
Now $\sin 80^{O} = \frac{m\overline{AC}}{m\overline{AB}}$

$0.9848 = \frac{m\overline{AC}}{5000}$

(using calculator)

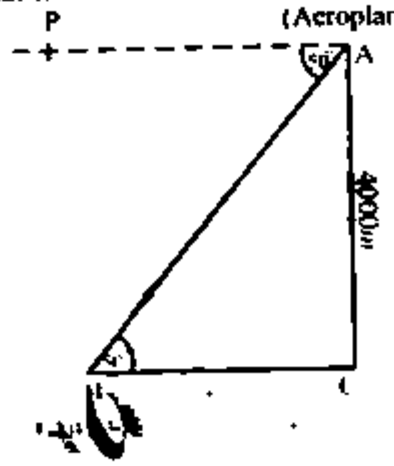
$m\overline{AC} = (0.9848)(5000)$

$= 4924$ meters



6. ***An aeroplane pilot flying at an altitude of 4000m wishes to make an approach to an airport at an angle of 50^0 with the horizontal. How far from the airport will the plane be when the pilot begins to descend?***

Solution: Let A be the point-4000 high when the pilot decides to descend making an angle of 50^O with the horizontal line.



Let $m\overline{BC} = x$ m be the distance of aeroplane from the airport in the horizontal direction.

Now $\tan 50 = \dfrac{4000}{x}$

$1.1918 = \dfrac{4000}{x}$ (using calculator)

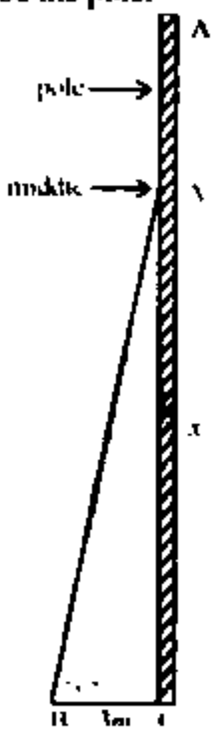
$x = \dfrac{4000}{1.1918}$

$= 3356.3$ m

A guy wire (supporting wire) runs from the middle of a utility pole to the ground. The wire makes an angle of 78.2^0 with the ground and touch the ground 3

meters from the base of the pole. Find the height of the pole.

Solution: Let $\overline{AC}$ be the pole.



(ii) M the middle of the pole.

(iii) $\overline{MB}$ be the guy wire.

(iv) $m\overline{BC} = 3$ m.

(v) $m\angle MBC = 78.2^0$.

$$\tan 78.2^0 = \frac{x}{3}$$

$$4.7867 = \frac{x}{3} \quad \text{(using calculator)}$$

$$x = 4.7867 \times 3$$
$$x = 14.3601$$
$$m\overline{MC} = 14.3601$$
$$2m\overline{MC} = 2 \times 14.3601$$

i.e. $m\overline{AC} = 28.7202$ m [M is in middle of AC]

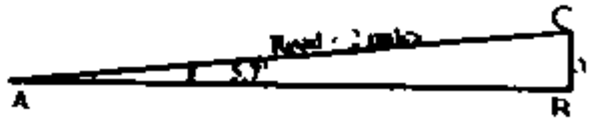
8. A road is inclined at an angle 5.7^{O}. Suppose that we drive 2 miles up this road starting from sea level. How high above sea level are we?

Solution: Let

(i) $\overline{AC} = 2$ miles be the road.

(ii) Height above sea level be $= x$ mile.

(iii) $m\angle BAC = 5.7^{O}$



Now $\frac{x}{2} = \sin 5.7^{O}$

$\frac{x}{2} = 0.0992$ (using calculator)

$x = 2(0.0992)$

$= 0.199$ miles.

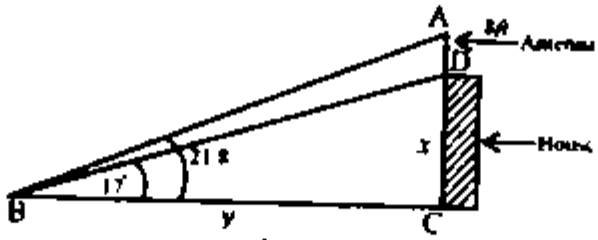
9. ***A television antenna of 8 feet height is located on the top a house. From a point on the ground the angle of elevation to the top of the house is 17^{O} and the angle of elevation to the top of the antenna is 21.8^{O}. Find the height of the house.***

Solution: Let

[illegible]

(ii) $m\overline{AD} = 8$ ft (height of antenna)

(iii) $m\angle DBC = 17^{\circ}$



$m\angle ABC = 21.8^{\circ}$

To find $m\overline{DC} = x$ (say)

Now $m\overline{AC} = (8 + x)$ feet.

$$\frac{m\overline{DC}}{m\overline{BC}} = \tan 17^{\circ}$$

$$\frac{x}{y} = .3057 \qquad \text{(i)}$$

$$\frac{m\overline{AC}}{y} = \tan 21.8^{\circ}$$

$$\frac{x+8}{y} = 0.3999 \qquad \text{(ii)}$$

$$y = \frac{x}{.3057} \quad \text{from (i)}$$

Putting $y = \frac{x}{.3057}$ in (ii)

$$\frac{x+8}{\frac{x}{.3057}} = 0.3999$$

$$x + 8 = \frac{x}{.3057} \times .3999$$

$$.3057(x+8) = .3999x$$
$$.3057x + 2.4456 = .3999x$$
$$2.4456 = .3999x - .3057x$$
$$2.4456 = 0.0942x$$
$$x = \frac{2.4456}{.0942}$$
$$= 25.96 \text{ feet.}$$

10. ***From an observation point, the angles of depression of two boats in line with this point are found to 30^O and 45^O. Find the distance between the two boats if the point of observation is 4000 feet high.***

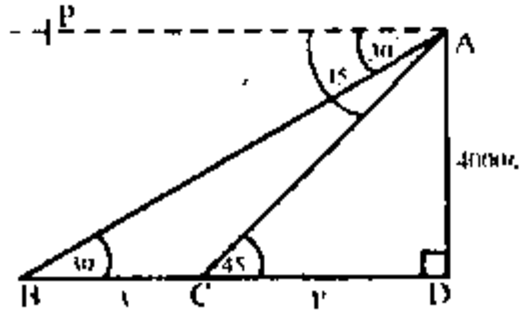
Solution: Let

(i) B and C be the places of boats.

(ii) A is point of observation.

(iii) $m\angle BAP = 30^0$

(iv) $m\angle CAP = 45^0$



To find $m\overline{BC} = x$ (say)

$m\overline{AD} = 4000$ ft. (given)

Now $m\angle ACD = 45^0$ (alternate of $m\angle PAC$)

and $m\angle ABD = 30^0$ (alternate of $m\angle PAB$)

In $\triangle ACD$, $\tan 45^0 = \dfrac{4000}{y}$

$$1 = \frac{4000}{y}$$

Thus $\quad y = 4000$ ft. $\quad$ (i)

In ΔABD

$$\tan 30^0 = \frac{4000}{y+x}$$

$$.5776 = \frac{4000}{4000+x} \quad \text{(using calculator) from (i)}$$

$$(4000+x).5776 = 4000$$

$$4000 \times .5776 + .5776x = 4000$$

$$2310.4 + .5776x = 4000$$

$$.5776x = 4000 - 2310.4$$

$$= 1689.6$$

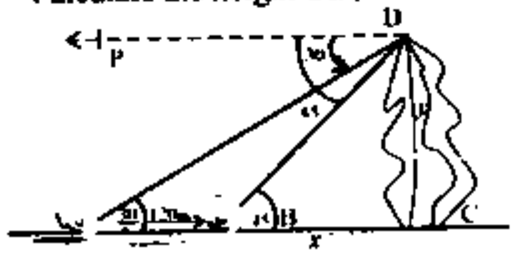
$$x = \frac{1689.6}{.5776}$$

$$= 2925.2 \text{ feet.}$$

11. ***Two ships, which are in line with the base of a vertical cliff, are 120 meters apart. The angles of depression from the top of the cliff to the ships are 30^O and 45^OC, as show in the diagram.***

(a) Calculate the distance BC

(b) Calculate the height CD, of the cliff.



Solution:

Let $m\overline{BC} = x$ m

$m\overline{CD} = y$ m

$m\angle CBD = m\angle PDB = 45^{O}$

$m\angle DAC = m\angle PDA = 30^{O}$

In ΔDCB

$$\frac{y}{x} = \tan 45^{O}$$

$$\frac{y}{x} = 1$$

$$\boxed{y = x} \quad \text{(i)}$$

In ΔDAC

$$\frac{y}{x+120} = \tan 30^{O}$$

$$\frac{y}{x+120} = 0.5772$$

$$\frac{x}{x+120} = 0.5772 \quad \text{(put } y = x \text{ from i)}$$

$$x = (x+120)(0.5772)$$

$$x = .5772x + 69.264$$

$$x - .5772x = 69.264$$

$$0.4228x = 69.264$$

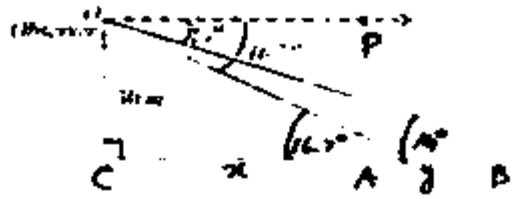
$$x = \frac{69.264}{.4228}$$

$$x = 163.822 \text{ m}$$

$$mCD = 163.822 \text{ m} \quad \text{as}(x = y)$$

Thus $m\overline{BC} = m\overline{CD} = 163.822$ *m.*

12. ***Suppose that we are standing on a bridge 30 feet above a river watching a log (piece of wood) floating toward we. If the angle with the horizontal to the front of the log is 16.7^{O} and angle with the horizontal to the back of the log is 14^{O}, how long is the log?***



Solution: Let,

(i) $m\overline{AB} = y$ m

(ii) $m\overline{CA} = x$ m

$m\angle CAO = 16.7^O$ (alternate to $\angle AOP = 16.7^O$)

$m\angle CBO = 14^O$ (alternate to $\angle BOP = 14^O$)

from ΔCAO

$$\frac{30}{x} = \tan 16.7$$

$$\frac{30}{x} = .3000$$

$$x = \frac{30}{.30}$$

$$x = 100 \text{ m} \quad \text{(i)}$$

From $\Delta\, CBO$

$$\frac{30}{x+y} = \tan 14^O$$

$$\frac{30}{x+y} = .2493$$

$$x+y = \frac{30}{.2493}$$

$$x+y = 120.3369$$

$$100+y = 120.3369$$

$$y = 120.3369 - 100 \quad (\text{from i, } x = 100)$$

$$y = 20.3369 \text{ m}$$

length of the log $= 20.3369$ m

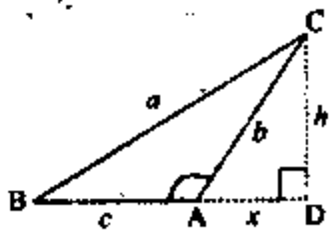
8 PROJECTION OF A SIDE OF A TRIANGLE

THEOREM 1

8.1(i) ***In an obtuse angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.***

Given: ABC is triangle having an obtuse angle BAC at A. Draw $\overline{CD}$ perpendicular on $\overline{BA}$ produced. So that $\overline{AD}$ is the projection of $\overline{AC}$ on $\overline{BA}$ produced.

Take $m\overline{BC} = a$, $m\overline{CA} = b$, $m\overline{AB} = c$, $m\overline{AD} = x$ and $m\overline{CD} = h$.



To prove: $\left(\overline{BC}\right)^2 = \left(\overline{AC}\right)^2 + \left(\overline{AB}\right)^2 + 2\left(m\overline{AB}\right)\left(m\overline{AD}\right)$

i.e., $a^2 = b^2 + c^2 + 2cx$

Proof:

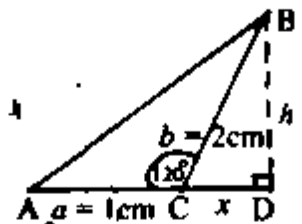
Statements	***Reasons***
In $\angle rt\ \Delta CDA$	
$m\angle CDA = 90°$	Given
$\therefore \left(\overline{AC}\right)^2 = \left(\overline{AD}\right)^2 + \left(\overline{CD}\right)^2$	Pythagoras Theorem
or $b^2 = x^2 + h^2$ (i)	
In $\angle rt\ \Delta CDB$,	
$m\angle CDB = 90°$	Given

$\therefore \quad (\overline{BC})^2 = (\overline{BD})^2 + (\overline{CD})^2$	Pythagoras Theorem $\overline{BD} = \overline{BA} + \overline{AD}$
or $\quad a^2 = (x + c)^2 + h^2$ $= c^2 + 2cx + x^2 + h^2$ (ii)	
Hence $a^2 = c^2 + 2cx + b^2$ *i.e.,* $\quad a^2 = b^2 + c^2 + 2cx$	Using (i) and (ii)
or $\quad (\overline{BC})^2 = (\overline{AC})^2 + (\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD})$	

EXERCISE 8.1

Q.1 ***Given $m\overline{AC} = 1cm$, $m\overline{BC} = 2cm$, $m\angle C = 120°$, Compute the length $\overline{AB}$ and the area of ΔABC, where $m\overline{CD} = m\overline{BC}\ Cos\ (180° - C)$***

(USE THEOREM 1)



$c^2 = a^2 + b^2 + 2ax$ (Theorem)

$c^2 = (1)^2 + (2)^2 + 2(1)(x)$ (i)

Now $m\overline{CD} = x \quad = \overline{BC}$ Cos (180 – C)

$= 2$ Cos (180 – 120)

$= 2$ Cos 60

$= 2 \times \frac{1}{2}$

$= 1$

Putting value of x in (i)

$C^2 = (1)^2 + (2)^2 + 2(1)(1)$

$= 1 + 4 + 2$

$$C^2 = 7$$
$$C = \sqrt{7}$$
$$C = 2.646 \text{ cm}$$

Thus $m\overline{AB} = 2.464\text{cm}.$

$$BD = h = \sqrt{(2)^2-(1)^2}$$
$$= \sqrt{4-1} = \sqrt{3}$$

Area of Δ ABC $= \frac{1}{2} \times AC \times BD$

$$= \frac{1}{2} \times 1 \times \sqrt{3}$$
$$= \frac{\sqrt{3}}{2}$$
$$h = \sqrt{3}$$
$$s = \frac{1}{2}\ AC \times BD$$
$$= \frac{1}{2}\ (1+1) \times \sqrt{3}$$
$$= \frac{\sqrt{3}}{2}$$

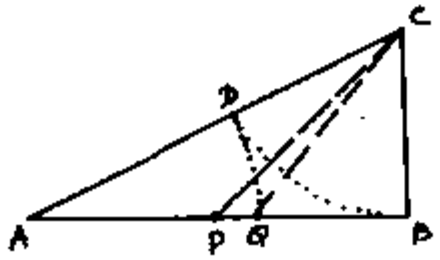
Q.2 ***If $\overline{AB}$ is bisected at P and also divided internally or externally at Q then prove that:***

$$(\overline{AQ})^2 + (\overline{BQ})^2 = 2\left[(\overline{AP})^2 + (\overline{PQ})^2\right]$$

Sol. In the figure $m\overline{AP} = m\overline{BP} = m\overline{BC} = m\overline{CD}$

$m\overline{AD} = m\overline{AQ}$

Because of a being mid point of $\overline{AB}$ and Q is the point of internal division.



Now, In ΔAPC, $\overline{PB}$ is projection of $\overline{CP}$

$\therefore \quad AC^2 = AP^2 + PC^2 + 2\,\overline{AP} \times \overline{PB}$

$= AP^2 + PC^2 + 2\,\overline{AP} \times \overline{AP}$ [as $\overline{AP} = \overline{PB}$]

$= AP^2 + PC^2 + 2AP^2$

$AC^2 = 3AP^2 + PC^2$ (i)

In ΔAQC, $\overline{QB}$ is projection of $\overline{CQ}$

$\therefore \quad AC^2 = AQ^2 + CQ^2 + 2\,\overline{AQ} \times \overline{QB}$ (ii)

and In ΔPQC

$PC^2 = PQ^2 + CQ^2 + 2\,\overline{PQ} \times \overline{QB}$ (iii)

$AC^2 = 3AP^2 + PQ^2 + CQ^2 + 2\,\overline{PQ} \times \overline{QB}$ from (i), (iii)

$3AP^2 + PQ^2 + CQ^2 + 2\,\overline{PQ} \times \overline{QB} = + AQ^2 + CQ^2 + 2\,\overline{AQ} \times \overline{QB}$

$3AP^2 + PQ^2 = AQ^2 + 2\,\overline{AQ} \times \overline{QB} - 2\,\overline{PQ} \times \overline{QB}$

$2AP^2 + AP^2 + 2PQ^2 - PQ^2 = AQ^2 + 2QB(AQ - PQ)$

$2AP^2 + 2PQ^2 = AQ^2 + PQ^2 - \overline{AP}^2 + 2QB(AQ-PQ)$

$2AP^2 + 2PQ^2 = AQ^2 + BQ^2 - \overline{AP}^2 + 2\overline{QB}\ \overline{AP}$

Thus $\overline{AQ}^2 + \overline{BQ}^2 = 2AP^2 + 2PQ^2$

$AQ^2 + BQ^2 = 2[(AP)^2 + (PQ)^2]$

THEOREM 2

8.1(ii) In any triangle, the square on the side opposite to acute angle is equal to sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.

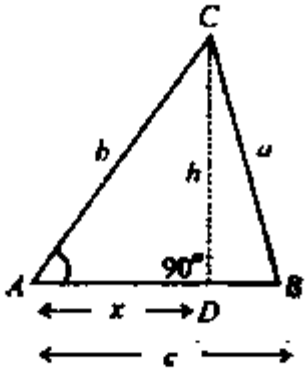
Given: ΔABC with an acute angel CAB at A.

Take $m\overline{BC} = a$, $m\overline{CA} = b$,

$m\overline{AB} = c$

Draw $\overline{CD} \perp \overline{BA}$ so that $\overline{AD}$ is projection of $\overline{AC}$ on $\overline{AB}$

Also, $m\overline{AD} = x$ and $m\overline{CD} = h$



To prove: $\left(\overline{BC}\right)^2 = \left(\overline{AC}\right)^2 + \left(\overline{AB}\right)^2 - 2\left(m\overline{AB}\right)\left(m\overline{AD}\right)$

i.e., $a^2 = b^2 + c^2 - 2cx$

Proof:

Statements	***Reasons***
In $\angle rt\ \Delta CDA$	
$m\angle CDA = 90°$	Given
$\therefore\ \left(\overline{AC}\right)^2 = \left(\overline{AD}\right)^2 + \left(\overline{CD}\right)^2$	Pythagoras Theorem
i.e., $b^2 = x^2 + h^2$ (i)	
In $\angle rt\ \Delta CDB,$	
$m\angle CDB = 90°$	Given
$\left(\overline{BC}\right)^2 = \left(\overline{BD}\right)^2 + \left(\overline{CD}\right)^2$	Pythagoras Theorem
$a^2 = (c-x)^2 + h^2$	From the figure
or $a^2 = c^2 - 2cx + x^2 + h^2$ (ii)	
$a^2 = c^2 - 2cx + b^2$	Using (i) and (ii)
Hence $a^2 = b^2 + c^2 - 2cx$	
or $\left(\overline{BC}\right)^2 = \left(\overline{AC}\right)^2 + \left(\overline{AB}\right)^2$	
$-2(m\overline{AB})(m\overline{AD})$	

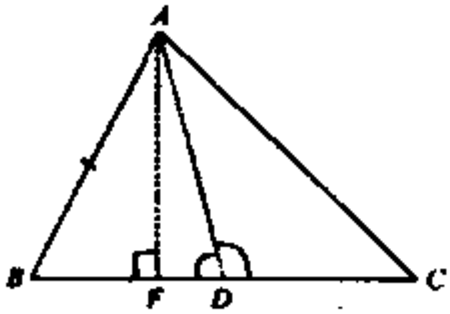
THEOREM 3

(APOLLONIUS' THEOREM)

8.1(iii) In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.

Given: In a ΔABC, the median $\overline{AD}$ bisects $\overline{BC}$.

i.e., $m\overline{BD} = m\overline{CD}$



To prove: $(\overline{BC})^2 + (\overline{AC})^2 = 2(\overline{BD})^2 + 2(\overline{AD})^2$

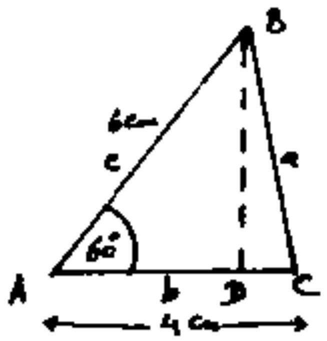
Construction: Draw $AF \perp BC$

Proof:

Statements	***Reasons***
In ΔADB	
Since $\angle ADB$ is acute at D	
$\therefore$ $(\overline{AB})^2 = (\overline{BD})^2 + (\overline{AD})^2$	Using Theorem 2
$2(m\overline{BD})(m\overline{FD})$ *(i)*	
Now in ΔADC since $\angle ADC$ is obtuse at D	
$\therefore$ $(\overline{AC})^2 = (\overline{CD})^2 + (\overline{AD})^2 + 2m\overline{CD} + m\overline{FD}$	Using Theorem 1
$(\overline{BD})^2 + (\overline{AD})^2 + 2m\overline{BD} + m\overline{FD}$ *(ii)*	
Thus $(\overline{AB})^2 + (\overline{AC})^2 = 2(\overline{BD})^2 + 2(\overline{AD})^2$	Adding (i) and (ii)

EXERCISE 8.2

Q.1 In a DABC calculate $m\overline{BC}$ when $m\overline{AB}$ = 6cm, $m\overline{AC}$ = 4cm and $m\angle A$ = 60°.



Calculation:

Let $\overline{BD} \perp \overline{AC}$

$$\frac{m\overline{AD}}{m\overline{AB}} = \cos 60^\circ$$

$$m\overline{AD} = m\overline{AB}\ \cos 60^\circ$$

$$= 6 \times \frac{1}{2}$$

$$m\overline{AD} = 3\text{cm} \quad \ldots\ldots (i)$$

Now in ΔABC, $m\angle A = 60^\circ$ (an acute angle)

Therefore, $a^2 = c^2 + b^2 - 2\,\overline{AD}\times\overline{AC}$ (Theorem)

$$= (6)^2 + (4)^2 - 2(3)(4)$$

$$= 36 + 16 - 24$$

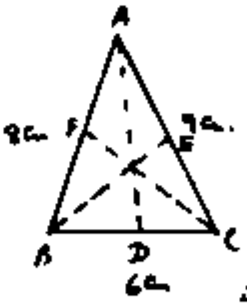
$$a^2 = 28$$

$$\therefore \quad a = \sqrt{28}$$

$$A = 5.29\text{cm}$$

Q.2 ***What are lengths of shortest and largest medians of a triangle whose sides are 6cm, 8cm and 9cm.***

Sol. Let ABC be the triangle in which $\overline{AD}$, BE and $\overline{CF}$ are the medians.



Applying Apollonius theorem.

$AB^2 + AC^2 = 2BD^2 + 2AD^2$

Putting value of $\overline{AB}, \overline{BD} = \frac{1}{2}\overline{BC} = 3$cm we get.

$(8)^2 + (9)^2 = 2(3)^2 + 2(AD)^2$

$64 + 81 = 18 + 2(AD)^2$

$2(AD)^2 = 64 + 81 - 18$

$2(AD)^2 = 127$

$(AD)^2 = \frac{127}{2}$

$(AD)^2 = 63.5$

$m\overline{AD} = \sqrt{63.5}$

$= 7.968$cm (i)

Now $AB^2 + BC^2 = 2(CE)^2 + 2(BE)^2$

Putting values

$(8)^2 + (6)^2 = 2\left(\frac{9}{2}\right)^2 + 2(BE)^2 \qquad \left[m\overline{CE} = \frac{1}{2}\overline{CA}\right]$

$64 + 36 = 2 \times \frac{81}{4} + 2(BE)^2$

$$100 = \frac{81}{2} + 2(BE)^2$$

$$2(BE)^2 = 100 - \frac{81}{2}$$

$$2(BE)^2 = \frac{200-81}{2}$$

$$= \frac{119}{2} = 59.5$$

$$2(BE)^2 = 59.5$$

$$(\overline{BE})^2 = \frac{59.5}{2}$$

$$= 29.75$$

$$\therefore \quad \overline{BE} = \sqrt{29.75}$$

$$= 5.454\text{cm.} \qquad \text{(ii)}$$

and $(BC)^2 + (CA)^2 = 2(\overline{CF})^2 + 2(\overline{AF})^2$

Putting values

$$(6)^2 + (9)^2 = 2(\overline{CF})^2 + 2\left(\frac{\overline{AF}}{2}\right)^2$$

$$36 + 81 = 2(\overline{CF})^2 + 2\times\left(\frac{8}{2}\right)^2$$

$$= 2(\overline{CF})^2 + 32$$

$$2(CF)^2 = 36 + 81 - 32$$

$$2(CF)^2 = 85$$

$$(CF)^2 = \frac{85}{2}$$

$$(CF)^2 = 42.5$$

$$CF = \sqrt{42.5}$$

$$m\overline{CF} = 6.519 \qquad \text{(iii)}$$

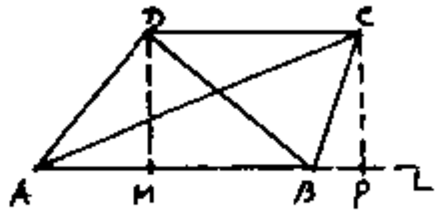
From (i), (ii), (iii)

Shortest length = 5.45cm

Longest length = 7.97cm

Q.3 ***In a quadrilateral ABCD, if $\overline{AB} \parallel \overline{CD}$, then prove that $(AC)^2 + (BD)^2 = (AD)^2 + (BC)^2 + 2\,\overline{AB} \times \overline{CD}$***

Sol. ***Given:***



(i) $\overline{AB} \parallel \overline{CD}$

(ii) A is joined to C

(iii) B is joined to D.

(iv) $\overline{AB}$ is extended towards B.

(v) $\overline{CP} \perp \overline{AB}$ extended.

(vi) $\overline{DM} \perp \overline{AB}$

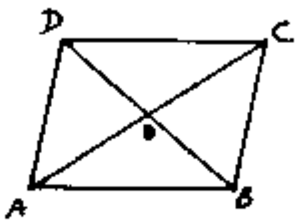
To prove: $(AC)^2 + (BD)^2 = (AD)^2 + (BC)^2 + \overline{AB} \times \overline{CD}$

Proof:

Statements	Reasons
In ΔABC, $\angle ABC$ is obtuse.	
$\therefore$ $AC^2 = AB^2 + BC^2 + 2\,\overline{AB} \times BP$ (i)	Theorem
In ΔABD $\angle BAD$ is acute,	
$\therefore$ $BD^2 = AD^2 + AB^2 - 2AB \times AM$ (ii)	
Adding (i), (ii)	
$AC^2 + BD^2 = AB^2 + BC^2 + AD^2 + AB^2 +$	
$2\,\overline{AB} \times BP - 2\,\overline{AB} \times AM$	
$= AD^2 + BC^2 + 2AB^2 + 2AB(BP - \overline{AM})$	
$= AD^2 + BC^2 + 2AB[\,\overline{AB} + BP - AM]$	
$AC^2 + BD^2 = AD^2 + BC^2 + 2\,\overline{AB} \times \overline{MP}$	Prove
$= AD^2 + BC^2 + 2\,\overline{AB} \times \overline{CD}$	

Q.4 Prove that the sum of the squares of the sides of a parallelogram is equal to sum of the squares of its diagonals.

Sol.



Given: (i) ABCD is a $\|^{m}$.

(ii) $\overline{AC}$, $\overline{BD}$ are its diagonals.

To prove:

$$m(\overline{AB})^2 + (m\overline{BC})^2 + (m\overline{CD})^2 + (m\overline{DA})^2 = (m\overline{AC})^2 + (m\overline{BD})^2$$

Construction:

Applying Apollonius theorem on ΔABD

$AB^2 + AD^2 = 2(\overline{OB})^2 + 2(\overline{AO})^2$ (i)

$AB^2 + BC^2 = 2(OB)^2 + 2(CO)^2$ (ii) [ΔABC]

$BC^2 + CD^2 = 2(\overline{CO})^2 + 2(\overline{OD})^2$ (iii) [ΔDBC]

$(CD)^2 + DA^2 = 2(DO)^2 + 2(AO)^2$ (iv) [ΔDAC]

Adding (i), (ii), (iii), (iv)

$$2(AB)^2 + 2(BC)^2 + 2(CD)^2 + 2(DA)^2$$

$$= 4(OB)^2 + 4(CO)^2 + 4(\overline{DO})^2 + 4(\overline{AO})^2$$

$$= 4\left[\frac{\overline{BD}}{2}\right]^2 + 4\left[\frac{\overline{CA}}{2}\right]^2 + 4\left[\frac{\overline{DB}}{2}\right]^2 + \left[\frac{\overline{CA}}{2}\right]^2$$

$$= (BD)^2 + (CA)^2 + (DB)^2 + (CA)^2$$

$$2[(AB)^2 + (BC)^2 + (CD)^2 + (DA)^2] = 2(CA)^2 + 2(BD)^2$$

Dividing by 2

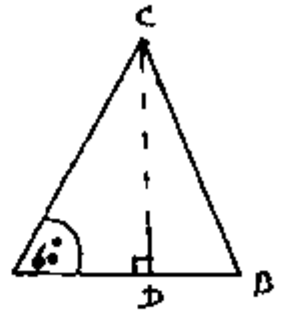
$$m(AB)^2 + (mBC)^2 + (mCD)^2 + (mDA)^2 = (mCA)^2 + (mBD)^2$$

Proved

MISCELLANEOUS EXERCISE – 8

Q.1 In a ΔABC, $m\angle A = 60°$. Prove that:

$$(BC)^2 = (AB)^2 + (AC)^2 - m\overline{AB} \times m\overline{AC}$$



Given: ABC is a triangle in which $m\angle A = 60°$

To prove: $(BC)^2 = (AB)^2 + (AC)^2 - m\overline{AB} \times m\overline{AC}$

Construction: Draw $\overline{CD} \perp \overline{AB}$

Proof:

Applying projection theorem, where $\overline{AD}$ is project of $\overline{AC}$ on AB

$$(BC)^2 = (AC)^2 + (AB)^2 - 2\overline{AB} \times \overline{AD}$$

Now $\dfrac{\overline{AD}}{\overline{AC}} = \text{Cos } 60°$

$$\overline{AD} = \overline{AC} \text{ Cos } 60$$

$$= (\overline{AC})\left(\frac{1}{2}\right)$$

$$m\overline{AD} = \frac{m\overline{AC}}{2}$$

Putting $m\overline{AD} = \dfrac{m\overline{AC}}{2}$ in (i), we get

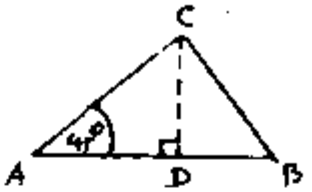
$$(BC)^2 = (AC)^2 + (AB)^2 - 2\overline{AB} \times \frac{\overline{AC}}{2}$$

$$(mBC)^2 = (m\overline{AC})^2 + (m\overline{AB})^2 - (m\overline{AB})(m\overline{AC})$$

Proved

Q.2 ***In a ΔABC, $m\angle A = 45°$, prove that***

$(BC)^2 = (AB)^2 + (AC)^2 - \sqrt{2}\, AB \times AC$



Given: In ΔABC

$m\angle A = 45°$

To prove: $(BC)^2 = (AB)^2 + (AC)^2 - \sqrt{2}\, \overline{AB} \times \overline{AC}$

Construction: Draw $CD \perp AB$

Now, applying projection theorem

$(BC)^2 = (AB)^2 + (AC)^2 - 2.\, AB \times \overline{AD}$ (i)

and $\dfrac{m\overline{AD}}{m\overline{AC}} = \text{Cos } 450°$

$m\overline{AD} = m\overline{AC} \times \dfrac{1}{\sqrt{2}}$

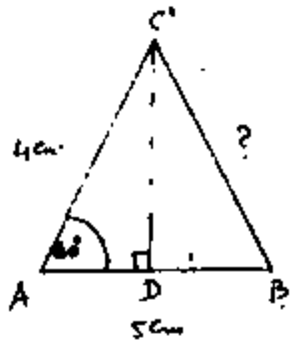
Putting value of $\overline{AD}$ in (i)

$(BC)^2 = (AB)^2 + (AC)^2 - 2\, AB \times \dfrac{AC}{\sqrt{2}}$

$(BC)^2 = (AB)^2 + (AC)^2 - \sqrt{2}\, AB \times \overline{AC}$

Proved.

Q.3 ***In a ΔABC, calculate*** $m\overline{BC}$ ***when*** $m\overline{AB} = 5cm$, $m\overline{AC} = 4cm$, $m\angle A = 60°$



Given: In $\triangle ABC$

$m\angle A = 60°$

$m\overline{AB} = 5cm$, $m\overline{AC} = 4cm$

To prove: To find $m\overline{BC}$

Constriction: Draw $\overline{CD} \perp \overline{AB}$

Applying projection theorem.

$$(BC)^2 = (AB)^2 + (AC)^2 - 2\overline{AB} \times \overline{AD}$$

Now, $\dfrac{m\overline{AD}}{m\overline{AC}} = \text{Cos } 60°$

$$m\overline{AD} = m\overline{AC}\ \text{Cos } 60°$$

$$= m\overline{AC} \times \frac{1}{2}$$

Putting value of $\overline{AD}$ in (i)

$$(BC)^2 = (AB)^2 + (AC)^2 - 2\ \overline{AB} \times \frac{\overline{AC}}{2}$$

$$= (AB)^2 + (AC)^2 - \overline{AB} \times \overline{AC}$$

Putting values of AB, AC.

$$= (5)^2 + (4)^2 - (5)(4)$$

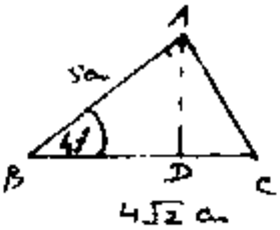
$$= 25 + 16 - 20$$

$$(BC)^2 = 21$$

$$m\overline{BC} = \sqrt{21}$$

$$= 4.472$$

Q.4 *In a* ΔABC, *calculate* $m\overline{AC}$ *when* $m\overline{AB} = 5cm$, $m\overline{BC} = 4\sqrt{2}$, $m\angle B = 45°$



Given: In ΔABC

$m\angle B = 45°$

$m\overline{AB} = 5cm$

$m\overline{BC} = 4\sqrt{2}$

To prove: To find $m\overline{AC}$

Construction: Draw $\overline{AD} \perp \overline{BC}$

Applying projection theorem.

$$(AC)^2 = (AB)^2 + (BC)^2 - 2\overline{BC} \times BD \ldots \quad (i)$$

Now, $\dfrac{m\overline{BD}}{m\overline{AB}} = \text{Cos}\, 45°$

$$m\overline{BD} = m\overline{AB} \times \text{Cos}\, 45°$$

$$= 5 \times \frac{1}{\sqrt{2}}$$

Putting value of AB , BC, BD in (i)

$$(AC)^2 = (5)^2 + (4\sqrt{2})^2 - 2 \times 4\sqrt{2} \times \frac{5}{\sqrt{2}}$$

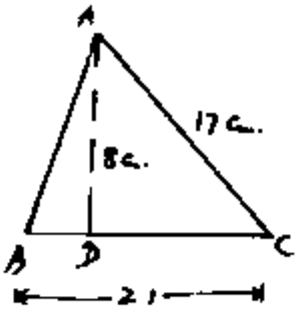
$$= 25 + 32 - 40 = 17$$

$$m\overline{AC} = \sqrt{17}$$

$$m\overline{AC} = 4.12cm$$

5 ***In a*** ΔABC, $m\overline{BC} = 21cm$, $m\overline{AC} = 17cm$,
$m\overline{AB}$ $10cm$.

Measure the length of projection of $\overline{AC}$ ***upon*** $\overline{BC}$.



iven: In ΔABC,

$m\overline{AB}$ $10cm$, $m\overline{AC}$ $17cm$,

$m\overline{BC}$ $21cm$,

$\overline{AD} \perp \overline{BC}$

$\overline{DC}$ is projection

$s = \dfrac{a+b+c}{2}$

$= \dfrac{10+17+21}{2}$

$= \dfrac{48}{2}$

$= 24$

$24 - 17 = 7$

$24 - 10 = 14$

$24 - 21 = 3$

Area of $\Delta ABC = \sqrt{(S)(S-a)(S-b)(S-c)}$

$= \sqrt{24 \times 7 \times 14 \times 3}$

$= \sqrt{7056}$

$= 84$ sq. cm.

Now Area of $\Delta ABC = \dfrac{\text{Base} \times \text{Alt}}{2}$

$$\frac{21 \times AD}{2} = 84$$

$$\overline{AD} = \frac{84 \times 2}{21}$$

$$= 8cm$$

Now, In ΔADC, $m\angle D = 90°$

By Pythagorean Theorem

$$(DC)^2 = (AC)^2 - (AD)^2$$

$$= (17)^2 - (8)^2$$

$$= 289 - 64$$

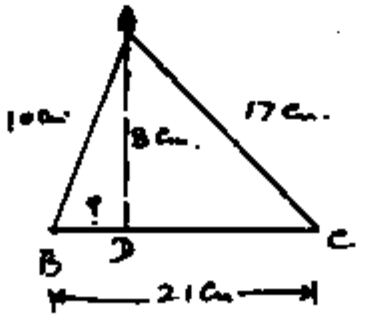
$$(DC)^2 = 225$$

$\therefore$ $m\overline{DC} = \sqrt{225}$

$$= 15cm.$$

Q.6 ***In*** ΔABC, $m\overline{BC}$ ***= 21cm,*** $m\overline{AC}$ ***= 17cm,*** $m\overline{AB}$ ***= 10cm.***

Calculate the projection of $\overline{AB}$ ***upon*** $\overline{BC}$.



Given: In ΔABC,

$m\overline{AB} = 10cm$, $m\overline{AC} = 17cm$,

$m\overline{BC} = 21cm$,

$\overline{AD} \perp \overline{BC}$

Required: To find $m\overline{BD}$

$$s = \frac{a+b+c}{2}$$
$$= \frac{10+17+21}{2}$$
$$= \frac{48}{2} = 24$$

$s-a = 24-17 = 7$
$s-b = 24-10 = 14$
$s-c = 24-21 = 3$

Area of $\Delta ABC = \sqrt{24 \times 7 \times 14 \times 3}$
$= \sqrt{7056}$
$= 84$ ......(i)

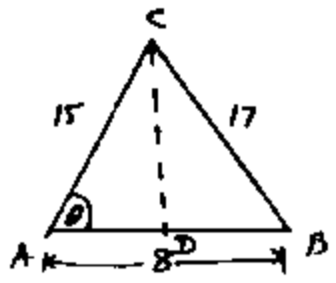
Now Area of $\Delta ABC = \dfrac{\text{Base} \times \text{Alt}}{2}$

$\dfrac{21 \times m\overline{AD}}{2} = 84$ from (i)

$m\overline{AD} = \dfrac{84 \times 2}{21}$
$= 8\text{cm}$

Now, In ΔABD,
$m\overline{BD} = \sqrt{(10)^2 - (8)^2}$
$= \sqrt{100-64}$
$= \sqrt{36}$
$m\overline{BD} = 6\text{cm}.$

Q.7 ***In a ΔABC, $a = 17cm$, $b = 15cm$ and $c = 8cm$, find $m\angle A$***



Given: In ΔABC

$a = 17cm$

$b = 15cm$

$c = 8cm$

Required: To find $m\angle A$

Calculation:

$$S = \frac{a+b+c}{2} = \frac{17+15+8}{2} = \frac{40}{2} = 20$$

$$\text{Area of } \Delta ABC = \sqrt{(s)(s-a)(s-b)(s-c)} = \sqrt{(20)(20-17)(20-15)(20-8)} = \sqrt{20\times3\times5\times12} = \sqrt{3600} = 60 \text{ sq.cm} \quad \ldots\ldots(i)$$

Now, $\text{Area of } \Delta ABC = \dfrac{\text{Base}\times\text{Alt}}{2}$

$$\frac{8\times CD}{2} = 60$$

$$CD = \frac{60\times2}{8} = 15cm.$$

Since $\overline{CA} = 15cm$ and

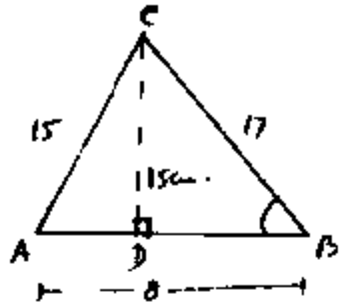
$\overline{CD} = 15cm$ Calculated

Therefore, $\overline{CA}$ coincides $\overline{CD}$ where as $\overline{CD}\perp\overline{AB}$

Thus, $\overline{CA}$ is also $\perp$ to $\overline{AB}$

In this way $m\angle A = 90°$

Q.8 In a ΔABC, $a = 17cm$, $b = 15$, $c = 8cm$ find $m\angle B$.



Given: In ΔABC

a $= 17$ cm

b $= 15$ cm

c $= 8$ cm

Calculations:

$$S = \frac{a+b+c}{2}$$

$$= \frac{15+17+8}{2}$$

$$= \frac{40}{2}$$

$$= 20$$

$$\text{Area of } \Delta ABC = \sqrt{(s)(s-a)(s-b)(s-c)}$$

$$= \sqrt{(20)(20-17)(20-15)(20-8)}$$

$$= \sqrt{20\times3\times5\times12}$$

$$= \sqrt{3600}$$

$$= 60 \text{ cm}^2 \quad \ldots\ldots\ldots \text{(i)}$$

Now, $$\text{Area of } \Delta ABC = \frac{\text{Base}\times\text{Alt}}{2}$$

$$= \frac{8\times CD}{2} \quad [\overline{CD} \perp AB]$$

$$\frac{8}{2}\times CD = 60 \text{ from (i)}$$

$$CD = \frac{60\times2}{8}$$

CD 15cm

Let m∠B θ

Sinθ $\frac{15}{17}$

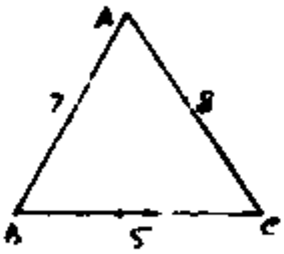
.8824

θ Sin^{-1} .8824

θ $(61.9)^{0}$ (Using calculator)

Q.9 Whether the triangle with sides 5cm, 7cm, 8cm is acute, obtuse or right angles.

Sol.



We check it as a right angled triangle.

Let 8 the hypotenuse.

Then $(8)^2$ $(7)^2 + (5)^2$

64 49 + 25

74

Which is not true. It is not a right angled triangle.

It is acute angled triangle, as the figure shows.

Q.10 Whether the triangle with sides 8cm, 15cm, 17cm is acute, obtuse or right angles.

Sol. We check it a right angled triangle.

Let 17 side be the hypotenuse.

By Pythagorean theorem.

$(17)^2$ $(15)^2 + (8)^2$

289 225 + 64

289 289

Which is true, therefore, the triangle is right angled triangle

9 CHORDS OF A CIRCLE

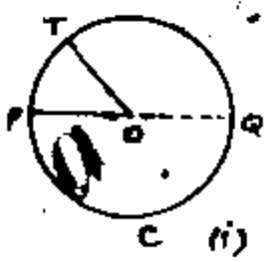
Basic Concepts of the Circle:

Circle: A circle is the locus of a moving point P in a plane which is always equidistant from some fixed point O.

In the affixed figure:

Line segment $\overline{PO}$ is radial segment.

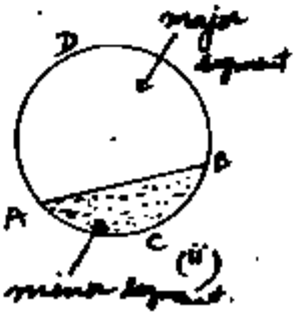
$\overline{PQ}$ is diameter of the circle. Circle line $PCQT$ is circumference of the circle.



In figure (ii)

(a) $\overline{AB}$ is chord of the circle.

(b) Minor and major segments are shown in it.



(c) In figure (iii)

Major and minor sectors are shown.

THEOREM 1

9.1(i) One and only one circle can pass through three non-collinear points.

Given:

A, B and C are three non collinear points in a plane.

To prove:

One and only one circle can pass through three non-collinear points A, B and C.

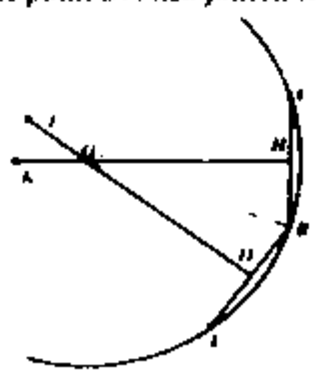
Construction:

Join A with B and B with C.

$\overline{AB}$ and $\overline{BC}$ are distinct and non collinear lines.

Draw $\overline{DF}$ ⊥ bisector to $\overline{AB}$ and $\overline{HK}$ ⊥ bisector to $\overline{BC}$.

So $\overline{DF}$ and $\overline{HK}$ are not parallel rather they meet each other at some point O. Also join A, B and C with point O.



Proof:

Statements	*Reasons*

Every point $\overline{DF}$ is equidistant from A and B.	$\overline{DF}$ ⊥ bisector to $\overline{AB}$ (construction)
In particular $m\overline{OA} = m\overline{OB}$ (i)	By Locus Theorem
Similarly every point on $\overline{HK}$ is equidistant from B and C.	$\overline{HK}$ is ⊥ bisector to $\overline{BC}$ (construction)
In particular $m\overline{OB} = m\overline{OC}$ (ii)	By Locus theorem
Now O is the only point common to $\overline{DF}$ and $\overline{HK}$ which is equidistant from A, B and C.	
i.e., $m\overline{OA}$ $m\overline{OB} = m\overline{OC}$	Using (i) and (ii)

however there is no such other point except O.
hence a circle with centre O and radius OA will pass through A, B and C.
Ultimately there is only one circle which passes through three given points A, B and C.

THEOREM 2

9.1(ii) A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.

Given:

M is the mid point of any chord $\overline{AB}$ of a circle with centre at O.

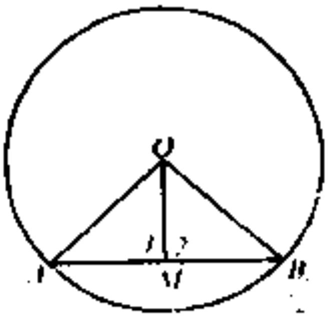
Where chord $\overline{AB}$ is not the diameter of the circle.

To prove:

$\overline{OM} \perp$ the chord $\overline{AB}$.

Construction:

Join A and B with centre O. Write $\angle 1$ and $\angle 2$ as shown in the figure.



Proof:

Statements	***Reasons***
In $\Delta OAM \leftrightarrow \Delta OBM$	
$m\overline{OA} = m\overline{OB}$	Radii of the same circle
$m\overline{AM} = m\overline{BM}$	Given
$m\overline{OM} = m\overline{OM}$	Common
$\therefore \quad \Delta OAM \cong \Delta OBM$	S.S.S ≅ S.S.S
$\Rightarrow \quad m\angle 1 = m\angle 2 \quad$ (i)	
i.e., $m\angle 1 + m\angle 2 = m\angle AMB = 180^{O}$ (ii)	Adjacent supplementary angles
$\therefore m\angle 1 = m\angle 2 = 90^{O}$	From (i) and (ii)
i.e., $\overline{OM} \perp \overline{AB}$	

THEOREM 3

9.1(iii) Perpendicular from the centre of a circle on a chord bisects it.

Given:

$\overline{AB}$ is the chord of a circle with centre at O so that $\overline{OM}$ $\perp$ chord $\overline{AB}$.

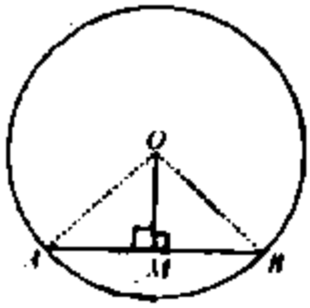
To prove:

M is the mid point of chord $\overline{AB}$

i.e., $m\overline{AM} = m\overline{BM}$

Construction:

Join A and B with centre O.



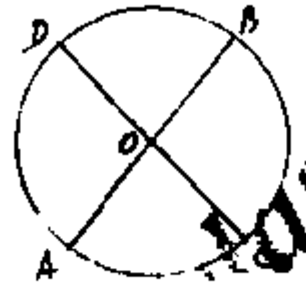
Proof:

Statements	Reasons
In $\angle rt\Delta^s OAM \leftrightarrow \Delta OBM$	
$m\angle OMA = m\angle OMB = 90^\circ$	Given
hyp. $\overline{OA}$ = hyp. $\overline{OB}$.	Radii of the same circle
$m\overline{OM} = m\overline{OM}$	Common
$\therefore$ $\Delta OAM \cong \Delta OBM$	In $\angle rt\Delta^s$ $H.S \cong H.S$
Hence, $m\overline{AM} = m\overline{BM}$	
$\Rightarrow$ $\overline{AM}$ *bisects* the chord $m\overline{AB}$	

EXERCISE 9.1

Q.1. Prove that, only the diameters of a circle are the intersecting chords which bisect each other.

Given: A circle with centre O and having $\overline{AB}$, $\overline{CD}$, that pass through the centre O of the circle.



To prove:

$\overline{AB}$ and $\overline{CD}$ diameters bisect each other.

Proof:

Statements	***Reasons***
$\overline{AB}$ is diameter that passes through centre O of three circle.	
$\therefore$ $m\overline{OA}$ $m\overline{OB}$ (i)	
Similarly $m\overline{OC}$ $m\overline{OD}$ (ii)	
Now $m\overline{OA}$ $m\overline{OC}$ (iii)	(Radii of the same circle)
From i, ii, iii	
mOA $m\overline{OB}$ $m\overline{OC}$ $m\overline{OD}$	

i.e: $\overline{AB}$ and CD are intersecting chords which bisect each other.

Q.2. ***Two chords of a circle do not pass through the centre. Prove that they cannot bisect each other.***

Given:

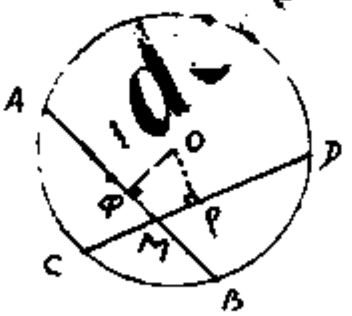
A circle with centre O, has $\overline{AB}$, $\overline{CD}$ chords that intersect each other at point M.

To prove:

M is not the mid-point of $\overline{AB}$, $\overline{CD}$

Construction:

Draw $\overline{OP} \perp \overline{CD}$, $\overline{OQ} \perp \overline{AB}$



Proof:

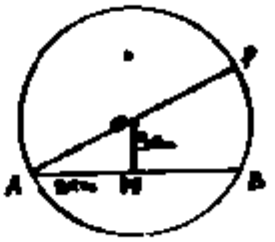
Statements	***Reasons***
O is centre of the circle $\overline{OP} \perp \overline{CD}$	Construct
Thus, $m\overline{CP}$ $m\overline{DP}$	Theorem
Now point M lies between C and P	
Therefore M is not mid-point of $\overline{CD}$	
Similarly M is not mid-point of $\overline{AB}$	

Hence, $\overline{AB}$ and $\overline{CD}$ cannot bisect each other.

Q.3. If length of the chord $\overline{AB}$ = 8cm. Its distance from the centre is 3cm, then measure the diameter of such circle.

Given:

O is the centre of a circle. $\overline{AB}$ is chord of the circle that $m\overline{AB} = 8$cm. $\overline{OM} \perp \overline{AB}$ that $m\overline{OM} = 3$cm.



Required:

To find the length of the diameter of the circle i.e; to find $m\overline{AP}$

Construction:

Join A to O and produce it to meet the circle at P.

Calculation:

M is the mid-point of $\overline{AB}$ (theorem)

$\therefore \quad m\overline{AM} = m\overline{BM} = \frac{8}{2} = 4\text{cm}.$

Now, In right angled triangle OAM by Pythagorean theorem.

$$(m\overline{AO})^2 = (m\overline{AM})^2 + (m\overline{OM})^2$$
$$= (4)^2 + (3)^2$$

$$= 16 + 9$$

$$(m\overline{AO})^2 = 25$$

$$\therefore \quad mOA = \sqrt{25}$$

$$= 5 \text{ cm.}$$

Now $m\overline{AP} = 2m\overline{AO}$

$$= 2 \times 5 = 10\text{cm.}$$

Q.4. Calculate the length of a chord which stands at a distance 5cm from the centre of a circle whose radius is 9cm.

Given:

O is centre of a circle in which

(i) $\overline{AB}$ is its chord.

(ii) $\overline{OM} \perp \overline{AB}$

(iii) $m\overline{OM} = 5$ cm

(iv) $m\overline{OB} = 9$ cm (radius)

Required:

To find $m\overline{AB}$

Calculation:

$\Delta\overline{OMB}$ is a right-angled triangle at M.

By Pythagorean theorem.

$$(m\overline{MB})^2 = (m\overline{OB})^2 - (m\overline{OM})^2$$
$$= (9)^2 - (5)^2$$
$$= 81 - 25$$
$$(m\overline{MB})^2 = 56$$
$$\therefore \quad m\overline{MB} = \sqrt{56}$$
$$= 7.48 \text{ cm.}$$
$$m\overline{AB} = 2 \times 7.48$$
$$= 14.96 \text{ cm.}$$

THEOREM 4

9.1(iv) If two chords of a circle are congruent then they will be equidistant from the centre.

Given:

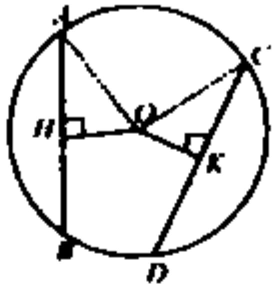
$\overline{AB}$ and $\overline{CD}$ are two equal chords of a circle with centre at O.

So that $\overline{OH} \perp \overline{AB}$ and $\overline{OK} \perp \overline{CD}$.

To prove: $m\overline{OH} = m\overline{OK}$

Construction:

Join O with A and O with C. So that we have $\angle rt\Delta^s OAH$ and OCK.



Proof:

Statements	Reasons
$\overline{OH}$ bisects chord $\overline{AB}$	$\overline{OH} \perp \overline{AB}$ By Theorem 3
i.e., $m\overline{AH} = \frac{1}{2} m\overline{AB}$ (i)	
Similarly $\overline{OK}$ bisects chord $\overline{CD}$	
i.e., $m\overline{CK} = \frac{1}{2} m\overline{CD}$ (ii)	
But $m\overline{AB} = m\overline{CD}$ (iii)	Given
Hence $m\overline{AH} = m\overline{CK}$ (iv)	Using (i), (ii) & (iii)
Now in $\angle rt\Delta^s\ OAH \leftrightarrow OCK$	Given $\overline{OH} \perp \overline{AB}$ and $\overline{OK} \perp \overline{CD}$
hyp $\overline{OA}$ = hyp $\overline{OC}$	Radii of the same circle
$m\overline{AH} = m\overline{CK}$	Already proved in (iv)
$\therefore$ $\Delta OAH \cong \Delta OCK$	H.S postulate
Hence $m\overline{OH} = m\overline{OK}$	

THEOREM 5

9.1(v) Two chords of a circle which are equidistant from the centre, are congruent.

Given:

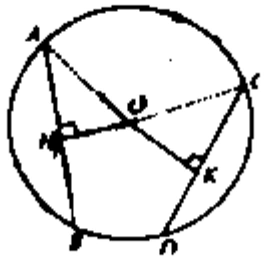
$\overline{AB}$ and $\overline{CD}$ are two chords of a circle with centre at O.

$\overline{OH} \perp \overline{AB}$ and $\overline{OK} \perp \overline{CD}$, so that $m\overline{OH} = m\overline{OK}$

To prove: $m\overline{AB} \quad m\overline{CD}$

Construction:

Join A and C with O. So that we can form $\angle rt\Delta^s$ OAH and OCK.



Proof:

Statements	***Reasons***
In $\angle rt\Delta^s OAH \leftrightarrow OCK$.	
hyp $\overline{OA}$ hypc $\overline{OC}$	Radii of the same circle.
$m\overline{OH} \quad m\overline{OK}$	Given
$\therefore \quad \Delta OAH \cong \Delta OCK$	H.S Postulate
So $m\overline{AH} \quad m\overline{CK}$ (i)	
But $m\overline{AH} \quad \frac{1}{2} m\overline{AB}$ (ii)	$\overline{OH} \perp$ chord $\overline{AB}$ (Given)

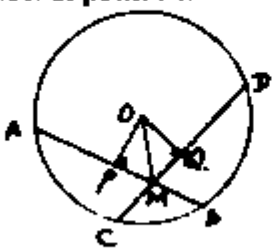
Similarly $m\overline{CK} = \frac{1}{2}m\overline{CD}$ (iii)	$\overline{OK}\perp$chord $\overline{CD}$ (Given)
Since $m\overline{AH} = m\overline{CK}$	Already proved in (i)
$\therefore \quad \frac{1}{2}m\overline{AB} = \frac{1}{2}m\overline{CD}$	Using (ii) & (iii)
or $\quad m\overline{AB} = m\overline{CD}$	

EXERCISE 9.2

Q.1. Two equal chords of a circle intersect, show that the segments of the one are equal corresponding to the segments of the other.

Given:

A circle with centre O, in which $m\overline{AB} = m\overline{CD}$, $\overline{AB}$ and $\overline{CD}$ intersect at point M.



Required:

$m\overline{MD} = m\overline{MA}$

$m\overline{MC} = m\overline{MB}$

Construction:

$\overline{OQ}\perp\overline{CD}$

$\overline{OP} \perp \overline{AB}$

Proof:

Statements	***Reasons***
In $\Delta^s OPM$ and OQM.	
$m\overline{OP} \quad m\overline{OQ}$	Theorem
$m\overline{OM} \quad m\overline{OM}$ (hypc).	Common
$\therefore \quad \Delta OPM \cong \Delta OQM$	
Thus $m\overline{PM} \quad m\overline{QM}$ (i)	
Now $m\overline{CQ} \quad m\overline{QD}$	Theorem
and $m\overline{AP} \quad m\overline{BP}$	
Since $m\overline{AB} \quad m\overline{CD}$	Given
$\therefore \frac{1}{2} m\overline{AB} \quad \frac{1}{2} m\overline{CD}$	
Thus $m\overline{AP} \quad m\overline{DQ}$ (ii)	
Adding (i), (ii)	
$m\overline{AP} + m\overline{PM} \quad m\overline{DQ} + m\overline{OM}$	
$\Rightarrow \quad m\overline{DM} \quad m\overline{AM}$ proved	Proved.
Now $m\overline{CD} - m\overline{MD} \quad m\overline{AB} - m\overline{AM}$	
Thus. $m\overline{CM} \quad m\overline{BM}$	Proved.

Q.3. As shown in the figure, find the distance between two parallel chords $\overline{AB}$ and $\overline{CD}$

Given:

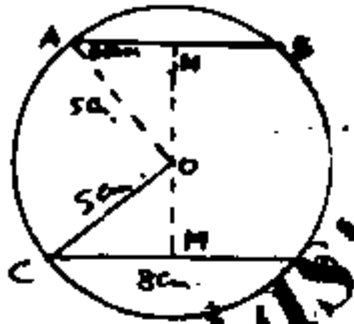
In the given figure.

$m\overline{AB} = 6$ cm

$m\overline{CD}$ 8 cm

$m\overline{OC}$ 5 cm

$\overline{OM} \perp \overline{CD}$, $\overline{ON} \perp \overline{MN}$



Required:

To find in $\overline{MN}$

$\overline{OM} \perp CD$

$\therefore \quad m\overline{CM} \quad \frac{8}{2} \quad 4$ cm

In rt. Angled triangle OCM.

$$(m\overline{OM})^2 = (m\overline{CO})^2 - (m\overline{CM})^2$$
$$= (5)^2 - (4)^2$$
$$= 25 - 16$$
$$(m\overline{OM})^2 = 9$$
$$\therefore \quad mOM = \sqrt{9}$$
$$3 \text{ cm.} \qquad \text{(i)}$$

Similarly in rt. Angled triangle OAN

$$(m\overline{ON})^2 = (m\overline{OA})^2 - (m\overline{AN})^2$$
$$= (5)^2 - (3)^2$$
$$= 25 - 9$$
$$(m\overline{ON})^2 = 16$$

$\therefore \quad m\overline{ON} = \sqrt{16}$

$= 4$ cm. (ii)

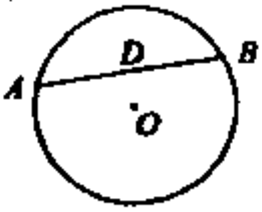
$m\overline{OM} + m\overline{ON} = 3 + 4 = 7$ cm

Thus distance between $\overline{AB}$, $\overline{CD}$ = 7 cm.

MISCELLANEOUS EXERCISE – 9

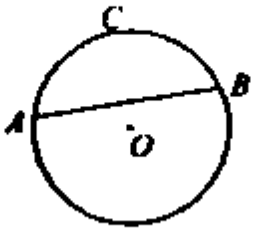
Q.1. Multiple Choice Questions.
Four possible answers are given for the following questions.
Tick (✓) the correct answer.

(i) In the circular figure, ADB is called



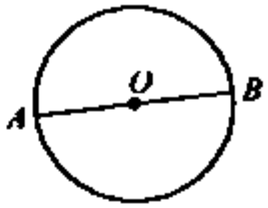
(a) an arc (b) a secant
(c) a chord (d) a diameter

(ii) In the circular figure, $\widehat{ABC}$ is called



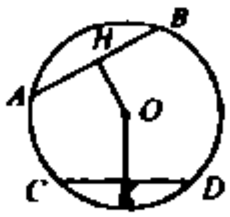
(a) an arc (b) a secant
(c) a chord (d) a diameter

(iii) In the circle figure, AOB is called



(a) an arc (b) a secant
(c) a chord (d) a diameter

(iv) In a circular figure, two chords AB and CD are equidistant from the centre. They will be



(a) parallel (b) non congruent
(c) congruent (d) perpendicular

(v) Radii of a circle are

(a) all equal (b) double of the diameter
(c) all unequal (d) half of any chord

(vi) A chord passing through the centre of a circle is called

(a) radius (b) diameter
(c) circumference (d) secant

(vii) Right bisector of the chord of a circle always passes through the

(a) radius (b) circumference
(c) centre (d) diameter

(viii) The circular region bounded by two radii and the corresponding arc is called

(a) circumference of a circle
(b) sector of a circle
(c) diameter of a circle
(d) segment of a circle

(ix) The distance of any point of the circle to its centre is called

(a) radius (b) diameter
(c) a chord (d) an arc

(x) Line segment joining any point of the circle to the centre is called

(a) circumference (b) diameter
(c) radial segment (d) perimeter

(xi) Locus of a point in a plane equidistant from a fixed point is called

(a) radius (b) circle
(c) circumference (d) diameter

(xii) The symbol for a triangle is denoted by

(a) $\angle$ (b) Δ
(c) $\perp$ (d) $\odot$

(xiii) A complete circle is divided into

(a) 90 degrees (b) 180 degrees
(c) 270 degrees (d) 360 degrees

(xiv) Through how many non collinear points, a circle can pass?

(a) one (b) two
(c) three (d) four

Answers:

(i)	c	(ii)	a	(iii)	d	(iv)	c	(v)	a
(vi)	b	(vii)	c	(viii)	b	(ix)	a	(x)	c
(xi)	b	(xii)	b	(xiii)	d	(xiv)	c		

Q.2 Differentiate between the following terms and illustrate them by diagrams.

(i) A circle and a circumference.

Ans. Circle:

A circle is the locus of a moving point which is always equidistant from a fixed point called its centre.

Circumference:

Boundary traced by the moving point is called circumference of the circle.

(ii) A chord and the diameter of a circle.

Ans. Chord:

A line segment joining any two points on the circumference is called chord of the circle.

Diameter:

A line segment joining any two points on the circumference and passing through its centre is diameter of the circle.

(iii) A chord and an arc f a circle.

Ans. Chord:

A line segment joining any two point on the circumference is called chord of the circle.

Arc: Part of the circumference is called an arc of the circle.

(iv) Minor arc and major arc of a circle.

Ans. Minor arc:

An arc smaller than half of the circumference is called minor arc of the circle.

Major arc:

An arc greater than half of the circumference is called major arc of the circle.

(v) Interior and exterior of a circle.

Ans. Interior of a Circle:

The area bounded by the circumference of the circle is its interior.

Exterior of a Circle:

The area outside the circumference of the circle is called exterior of the circle.

(vi) A sector and a segment of a circle.

Ans. Sector:

A sector of a circle is the plane figure bounded by two radii and the arc intercepted between them.

Segment:

A segment is the portion of a circle bounded by an arc and a corresponding chord.

10 TANGENT TO A CIRCLE

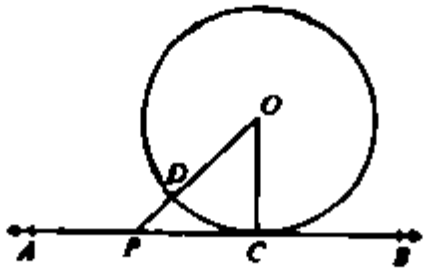
Secant:

A secant is straight line which cuts the circumference of a circle in two distinct points.

Tangent:

A tangent to a circle is a straight line which touches the circumference of the circle at a single point.

10.1(i) If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.



Given:

A circle with centre O and $\overline{OC}$ is the radial segment. $\overleftrightarrow{AB}$ is perpendicular to $\overline{OC}$ at its outer end C.

To prove:

$\overleftrightarrow{AB}$ is a tangent to the circle at C.

Construction:

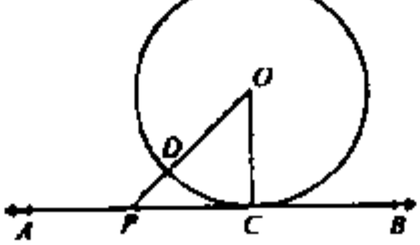
Take any point P other than C on $\overleftrightarrow{AB}$. Join O with P.

Proof:

Statements	*Reasons*
In ΔOCP,	
$m\angle OCP = 90°$	$\overleftrightarrow{AB} \perp \overline{OC}$ (given)
and $m\angle OPC < 90°$	Acute angle of right angled triangle.
$m\overline{OP} > m\overline{OC}$	Greater angle has greater side opposite to it.
$\therefore$ P is a point outside the circle.	$\overline{OC}$ is the radial segment.
Similarly, every point on $\overleftrightarrow{AB}$ except C lies outside the circle.	
Hence $\overleftrightarrow{AB}$ intersects the circle at one point C only.	
i.e., $\overleftrightarrow{AB}$ is a tangent to the circle at one point only.	

THEOREM 2

10.1(ii) The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.



Given:

In a circle with centre O and radius $\overline{OC}$.

Also a line segment $\overline{AB}$ is the tangent to the circle at point C.

To prove:

Tangent line $\overleftrightarrow{AB}$ and radial segment $\overline{OC}$ are perpendicular to each other.

Construction:

Take any point P other than C on the tangent line $\overleftrightarrow{AB}$.

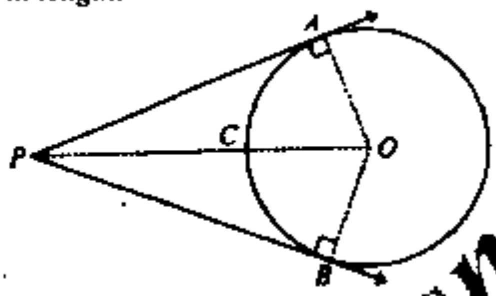
Join O with P so that $\overline{OP}$ meets the circle at D.

Proof:

Statements	***Reasons***
Line $\overleftrightarrow{AB}$ is the tangent to the circle at point C. Whereas $\overline{OP}$ cuts the circle at D.	Given and construction
$\therefore$ m$\overline{OC}$ = m$\overline{OD}$ (i)	Radii of the same circle
But m$\overline{OD}$ < m$\overline{OP}$ (ii)	Point P is outside the circle.
$\therefore$ m$\overline{OC}$ < m$\overline{OP}$	Using (i) and (ii)
So radius $\overline{OC}$ is shortest of all lines that can be drawn from O to the tangent line $\overleftrightarrow{AB}$	
Also $\overline{OC} \perp \overleftrightarrow{AB}$	Perpendicular line segment
Hence, radial segment $\overline{OC}$ is perpendicular to the tangent line $\overleftrightarrow{AB}$.	$\overline{OC}$ is the shortest from O to the tangent line $\overleftrightarrow{AB}$.

THEOREM 3

10.1(iii) Two tangents drawn to a circle from a point outside it, are equal in length.



Given:

Two tangents $\overline{PA}$ and $\overline{PB}$ are drawn from an external point P to the circle with centre O.

To prove:

$m\overline{PA} = m\overline{PB}$

Construction:

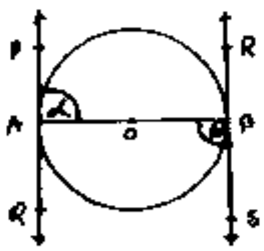
Join O with A, O with B and O with P, so that we form $\angle rt\Delta^s$ OAP and OBP.

Proof:

Statements	Reasons
In $\angle rt\Delta^s$ $OAP \leftrightarrow OBP$	
$m\angle OAP = m\angle OBP = 90^\circ$	Radii $\perp$ to the tangents $\overline{PA}$ and $\overline{PB}$
hyp. $\overline{OP}$ = hyp. $\overline{OP}$	Common
$m\overline{OA} = m\overline{OB}$	Radii of the same circle.
$\therefore$ $\Delta OAP \cong \Delta OBP$	In $\angle rt\Delta^s$ $H.S \cong H.S$
Hence, $m\overline{PA} = m\overline{PB}$	

EXERCISE 10.1

Q.1 Prove that the tangents drawn at the end of a diameter in a given circle must be parallel and conversely.



Given:

(i) A circle with centre O, has $\overline{AB}$ as diameter.

(ii) $\overleftrightarrow{PAQ}$ is tangent at point A.

(iii) $\overleftrightarrow{RBS}$ is tangent at point B.

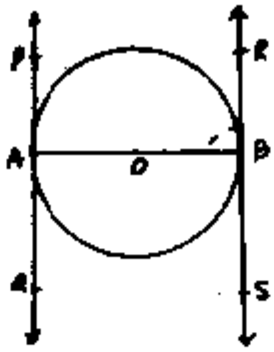
To prove:

$\overleftrightarrow{PAQ} \parallel \overleftrightarrow{RBS}$

Proof:

Statements	Reasons
$\overline{PA} \perp \overline{AB}$	Tangent
$\therefore \quad m\angle\alpha = 90^\circ \quad$ (i)	
$\overline{BS} \perp \overline{AB}$	Tangent
$\therefore \quad m\angle\theta = 90^\circ \quad$ (ii)	
Thus, $m\angle\alpha = m\angle\theta$	From (i), (ii)
Therefore, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$	Alternate angles are equal in measurement

Conversely:



Given:

A circle with centre O. $\overline{AB}$ is diameter of the circle. $\overleftrightarrow{PQ}$ touches the circle at A and $\overleftrightarrow{RS}$ touches the circle at B, such that $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$

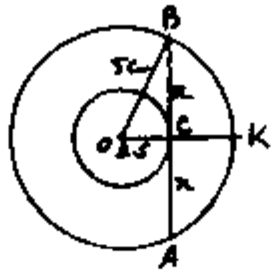
To prove:

$\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ are tangents.

Proof:

Statements	Reasons
$\overleftrightarrow{PQ}$ touches the circle at A, therefore all the points A on $\overleftrightarrow{PQ}$ lie out side the circle.	
Hence $\overline{OA} \perp \overleftrightarrow{PQ}$	Theorem
Thus $\overleftrightarrow{PQ}$ is tangent to the circle.	
Similarly $\overleftrightarrow{RS}$ is tangent to the circle.	

Q.2 ***The diameters of two concentric circles are 10cm and 5cm respectively. Look for the length of any chord of the outer circle which touches the inner one.***



Solution:

Let $\overline{AB}$ be any chord of the outer circle that touches the inner circle.

Join O to B and O to C.

Now OCB is a right angled triangle at angle C.

In BOC triangle.

$$m\overline{OB} = \frac{10}{2} = 5\text{cm}$$

$$m\overline{OC} = \frac{5}{2} = 2.5$$

Applying Pythagorean Theorem

$$(mBC)^2 = (mOB)^2 - (mOC)^2$$

$$m\overline{BC} = \sqrt{(5)^2 - (2.5)^2}$$

$$m\overline{BC} = \sqrt{25 - 6.25}$$

$$m\overline{BC} = \sqrt{18.75}$$

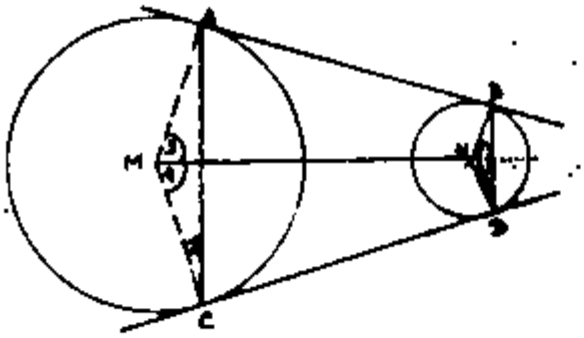
$$= 4.33 \text{ cm}$$

Now $m\overline{AB} = 2(m\overline{BC})$

$$= 2(4.33)$$

$$= 8.66 \text{ cm}$$

.3 ***$\overline{AB}$ and $\overline{CD}$ are the common tangents drawn to the pair of circles. A and C are the points of tangency of 1st circle and B and D are points of tangency of 2nd circle, then prove that $\overline{AC} \parallel \overline{BD}$.***



Given:

Two circles with centre M and N.

$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ their common tangents.

A is joined with C and B is joined with D.

To prove:

Prove that $\overline{AC} \parallel \overline{BD}$

Construction:

(i) Join M with A, C

(ii) Join N with B, D

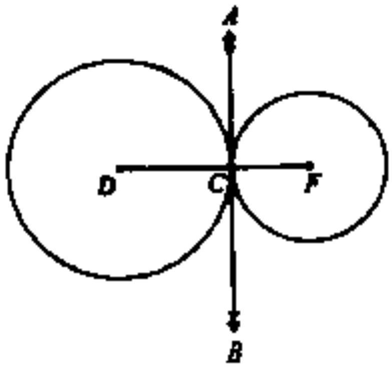
Proof:

Statements	***Reasons***
$\overline{MA} \perp \overline{AB}$ (i)	Theorem
$\overline{NB} \perp \overline{AB}$ (ii)	Theorem
Therefore, $\overline{MA} \parallel \overline{NB}$	From i, ii
Also $m\angle 3 = m\angle 1$ (iii)	Corresponding angles
Similarly $m\angle 4 = m\angle 2$ (iv)	
$m\angle 3 + m\angle 4 = m\angle 1 + m\angle 2$	From (iii) + (iv)
or $m\angle AMC = m\angle BND$	
Now in Δ^s AMC, BND	

$\frac{m\overline{MA}}{m\overline{NB}} = \frac{m\overline{MC}}{m\overline{ND}}$ and included angles $\angle AMC$, $\angle BND$ are congruent, therefore, Δ^s AMC, BND are similar. Thus, $m\angle MCA$ i.e., α and $m\angle NDB$ are equal. Hence $\overline{AC} \parallel \overline{BD}$	

THEOREM 4(A)

10.1(iv) If two circles touch externally then the distance between their centres is equal to the sum of their radii.



Given:

Two circles with centres D and F respectively touch each other externally at point C. So that $\overline{CD}$ and $\overline{CF}$ are respectively the radii of the two circles.

To prove:

(i). Point C lies on the join of centres D and F.

(ii) $m\overline{DF} = m\overline{DC} + m\overline{CF}$

Construction:

Draw $\overline{ACB}$ as a common tangent to the pair of circles at C.

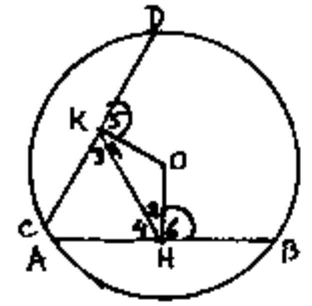
Join C with D and C with F.

Proof:

Statements	***Reasons***
Both circles touch externally at C whereas CD is radial segment and ACB is the common tangent.	
$\therefore$ $m\angle ACD = 90^\circ$ (i)	Radial segment $\overline{CD} \perp$ the tangent line $\overleftrightarrow{AB}$
Similarly CF is radial segment and ACB is the common tangent	
$\therefore$ $m\angle ACF = 90^\circ$ (ii)	Radial segment $\overline{CF} \perp$ the tangent line $\overleftrightarrow{AB}$
$m\angle ACD + m\angle ACF = 90^\circ + 90^\circ$	Adding (i) and (ii)
$m\angle DCF = 180^\circ$ (iii)	Sum of supplementary adjacent angles
Hence DCF is a straight line with point C between D and F	
So that $m\overline{DF} = m\overline{DC} + m\overline{CF}$	

EXERCISE 10.2

Q.1 ***$\overline{AB}$ and $\overline{CD}$ are two equal chords in a circle with centre O. H and K re respectively the mid points of the chords. Prove that HK makes equal angles with $\overline{AB}$ and $\overline{CD}$.***



Given:

(i) A circle with centre at O.

(ii) $m\overline{AB} = m\overline{CD}$

(iii) H is joined with K.

To prove:

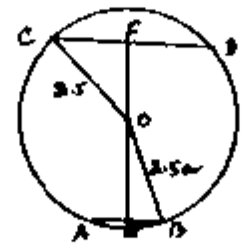
(i) $m\angle AHK = m\angle CKH$

(ii) $m\angle BHK = m\angle DKH$

Proof:

Statements	***Reasons***
In ΔHOK	
$m\overline{OH} = m\overline{OK}$	Theorem
$\therefore\ m\angle 1 = m\angle 2$ (i)	
And $m\angle 5 = m\angle 6$ (ii)	Each 90°
$m\angle 1 + m\angle 5 = m\angle 2 + m\angle 6$	From (i) + (ii)
Thus, $m\angle BHK = m\angle DKH$	Proved
$m\angle AHO = m\angle CKO$ (iii)	Each 90°
$m\angle 2 \quad m\angle 1$ (iv)	
$m\angle AHO - m\angle 2 = m\angle CKO - m\angle 1$	From (iii) – (iv)
$m\angle AHK = m\angle CKH$	Proved

Q.2 The radius of a circle is 2.5cm. $\overline{AB}$ and $\overline{CD}$ are two chords 3.9cm apart. If $m\overline{AB} = 1.4$cm, then measure the other chord.



Given:

O is the center of a circle.

(i) $m\overline{OB} = m\overline{OC} = 2.5\text{cm}$

$m\overline{AB} = 1.4\text{cm}$

$m\overline{EOF} = 3.9\text{cm}$

Required:

To measure $\overline{CD}$.

Construction:

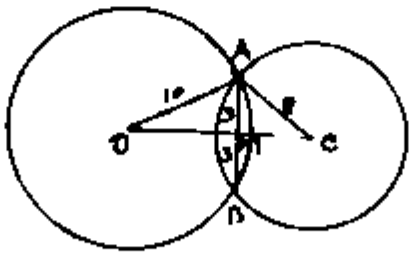
Join O with B and C.

Proof:

Statements	Reasons
In ΔOEB (a right angled Δ)	
$m\overline{OB} = 2.5\text{cm}$	Given
$m\overline{EB} = \frac{1}{2}m\overline{AB} = \frac{1}{2}(1.4)$	
$= .7\text{cm}$	
$(m\overline{OE})^2 = (m\overline{OB})^2 - (m\overline{EB})^2$	
$= (2.5)^2 - (.7)^2$	
$= 6.25 - .49$	
$(m\overline{OE})^2 = 5.76$	
$m\overline{OE} = \sqrt{5.76}$	
$= 2.4\text{cm}$	
Now $m\overline{OF} = m\overline{EF} - m\overline{OE}$	
$= 3.9 - 2.4$	
$= 1.5\text{cm}$ …………	(i)
In right angled triangle	

OCF	
$(m\overline{CF})^2 = (m\overline{OC})^2 - (m\overline{OF})^2$	
$= (2.5)^2 - (1.5)^2$	From (i)
$6.25 - 2.25$	
$(m\overline{CF})^2 = 4$	
$\therefore\ m\overline{CF} = 2$	(ii)
$m\overline{CD} = 2(m\overline{CF})$	
$= 2 \times 2$	From (ii)
$= 4cm$	

Q.3 ***The radii of two intersecting circles are 10cm, 8cm. If the length of their common chord is 6cm then find the distance between the centres.***



Calculations:

$\overline{AB} \perp \overline{OC}$ and M is their point of intersection.

In ΔOMA

$m\overline{OA} = 10cm$

$m\overline{AM} = 3cm.$ as $m\overline{AB} = 6cm$

$\therefore$ By Pythagorean Theorem

$m\overline{OM} = \sqrt{(m\overline{OA})^2 - (m\overline{AM})^2}$

$= \sqrt{(10)^2 - (3)^2}$

$= \sqrt{100 - 9}$

$= \sqrt{91}$

$= 9.54cm$

In ΔAMC

$m\overline{CA} = 8\text{cm}$

$m\overline{AM} = 3\text{cm}$

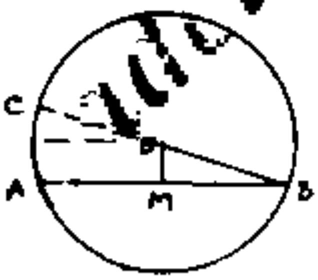
By Pythagorean Theorem

$$m\overline{MC} = \sqrt{(m\overline{CA})^2 - (m\overline{AM})^2}$$
$$= \sqrt{(8)^2 - (3)^2}$$
$$= \sqrt{64 - 9}$$
$$= \sqrt{55}$$
$$= 7.42$$

Thus, $m\overline{OC} = m\overline{OM} + m\overline{MC}$
$$= 9.54 + 7.42$$
$$= 16.96 \text{ cm}$$

Distance between the centres = 16.96cm

Q.4 Show that greatest chord in a circle is its diameter.



Given:

Let O be the centre of the circle.

$\overline{COB}$ is any diameter.

Let $\overline{AB}$ be any chord of the circle.

To prove:

$m\overline{CB} > m\overline{AB}$

Calculations:

Draw $\overline{OM} \perp \overline{AB}$

In right angled triangle OMB.

By Pythagorean Theorem

$$(m\overline{OB})^2 = (m\overline{OM})^2 + (m\overline{MB})^2$$

This means $m\overline{OB} > m\overline{MB}$

$\therefore\ 2(m\overline{OB}) > 2(m\overline{MB})$

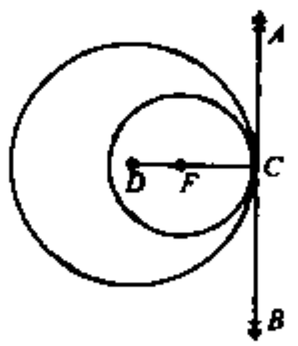
As $2(m\overline{OB})$ is length of the diameter and $2(m\overline{MB})$ is length of the chord $\overline{AB}$

Thus, $m\overline{CB} > m\overline{AB}$

Similarly it can be proved that diameter of a circle is its greatest chord.

THEOREM 4(B)

10.1(v) If two circles touch other internally, then the point of contact lies on the straight line through their centres and distance between their centres is equal to the difference of their radii.



Given:

Two circles with centres D and F touch each other internally at point C. So that $\overline{CD}$ and $\overline{CF}$ are the radii of two circles.

To prove:

(i) Point C lies on the join of centres D and F extended.

(ii) $m\overline{DF} = m\overline{DC} - m\overline{CF}$

Construction:

Draw $\overline{ACB}$ as the common tangent to the pair of circles at C.

Proof:

Statements	**Reasons**
Both circles touch internally at C whereas $\overleftrightarrow{ACB}$ is the common tangent and $\overline{CD}$ is the radial segment of the first circle.	
$\therefore \quad m\angle ACD = 90^{\circ} \quad$ (i)	Radial segment $\overline{CD} \perp$ the tangent line $\overline{AB}$
Similarly $\overline{ACB}$ is the common tangent and $\overline{CF}$ is the radial segment of the second circle.	
$\therefore \quad m\angle ACF = 90^{\circ} \quad$ (ii)	Radial segment $\overline{CF} \perp$ the tangent line $\overline{AB}$.
$\Rightarrow \quad m\angle ACD = m\angle ACF = 90^{\circ}$ Where $\angle ACD$ and $\angle ACF$ coincide each other with point F between D and C. Hence $m\overline{DC} = m\overline{DF} + m\overline{FC}$ (iii) i.e., $m\overline{DC} - m\overline{FC} = m\overline{DF}$ or $m\overline{DF} = m\overline{DC} - m\overline{FC}$	Using (i) and (ii)

EXERCISE 10.3

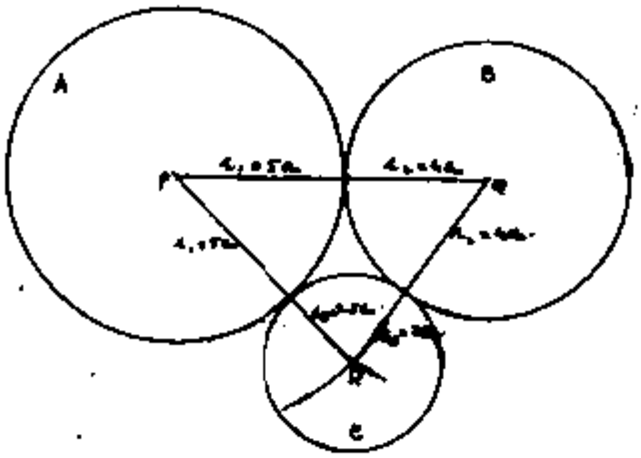
Q.1 ***Two circles with radii 5cm and 4cm touch each other externally. Draw another circle with radius 2.5cm touching the first pair, externally.***

Solution:

Let A, B, C be the circles and their radii be r_1, r_2, r_3 respectively.

$r_1 = 5\text{cm}$

$r_2 = 4cm$
$r_3 = 2.5cm$
$r_1 + r_2 = 5 + 4 = cm$
$r_1 + r_3 = 5 + 2.5$ cm
$r_2 + r_3 = 4 + 2.5$ cm



Steps of construction:

(i) Draw a line segment $\overline{PQ}$ of $5 + 4 = 9$cm long.

(ii) Take P centre and a circle of radius 5cm.

(iii) Take Q as centre and a circle of radius 4cm.

(iv) Take P as centre and draw an arc of radius $5 + 2.5 = 7.5$cm.

(v) Take Q as centre and draw an arc of radius $4 + 2.5 = 6.5$cm, this arc intersects the first arc at point R.

(vi) Take R as centre and draw a circle of radius 2.5cm. this circle touches externally the circles with centre P and Q.

Q.2 If the distance between the centres of two circles is the sum or the difference of their radii they will touch each other.

Solution:

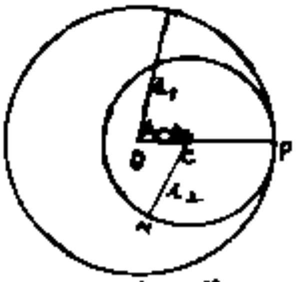
Steps of construction:

Fig (i)

Externally
fig (i)

(i) Let the two circles be with O and C as centres. Let their radii be r_1 and r_2. Draw the two centres with $m\overline{OC}$ equal to $r_1 + r_2$ apart. These circles touch each other externally at point P.

Fig (ii)



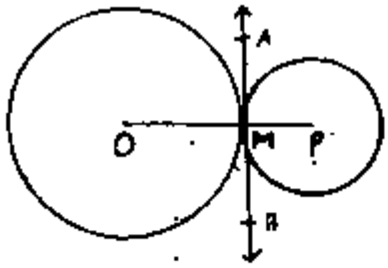
Internally
Fig (ii)

(i) Let the two circles be with O and C as centres. Let their radii be r_1 and r_2.

Take $m\overline{OC} = r_1 - r_2$ apart and draw the circles.

These circles touch each other internally at point P.

Q.3 ***The point of contact of two circles will be the point lying on the line of centres.***



Given:

The point of contact of circles with centres O and P is M.

To prove:

Point of contact M lies on the line joins the centres.

Construct:

Join M to O and P.

Proof:

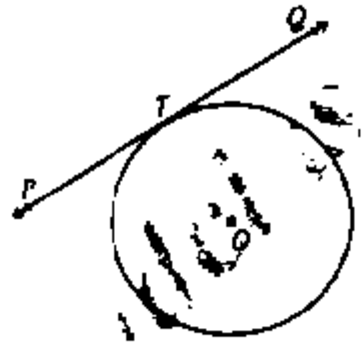
Statements	Reasons
OM is radical segment and $\overline{MA}$ tangent to the circle with centre O.	
$\therefore \quad \overline{OM} \perp \overline{MA}$	
i.e., $m\angle OMA = 90^{O}$ (i)	
$m\angle PMA = 90^{O}$ (ii)	
Similarly	
$m\angle OMA + m\angle PMA = 90^{O} + 90^{O} = 180^{O}$	From (i) + (ii)
Thus, $\overline{OMP}$ is a straight line segment	

Thus, point M lies on the line of centres of the circles.

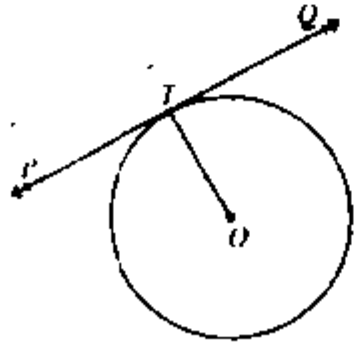
MISCELLANEOUS EXERCISE – 10

I. ***Multiple Choice Questions***
Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) In the adjacent figure of the circle, the line $\overline{PTQ}$ is named as
(a) an arc (b) a chord
(c) a tangent (d) a secant

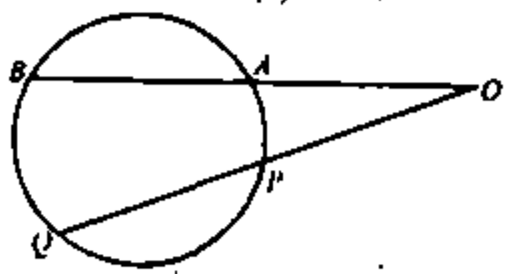


(ii) In a circle with centre O, if $\overline{OT}$ is the radial segment and $\overleftrightarrow{PTQ}$ is the tangent line, then
(a) $\overline{OP} \perp \overline{PQ}$ (b) $\overline{OT} \measuredangle \overline{PQ}$
(c) $\overline{OT} \parallel \overline{PQ}$ (d) $\overline{OT}$ is right bisector of $\overline{PQ}$



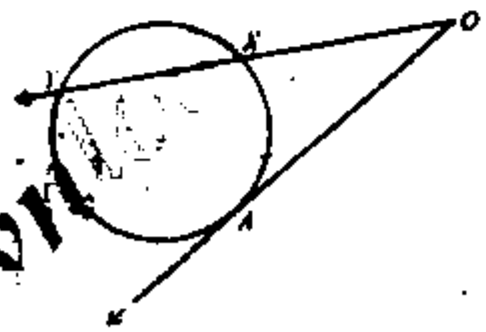
(iii) In the given diagram find $m\overline{OA}$ if $m\overline{OB}$ = 8cm. $m\overline{OP}$ 4cm and $m\overline{OQ}$ = 12cm

(a) 2cm (b) 2.67cm
(c) 2.8cm (d) 3cm



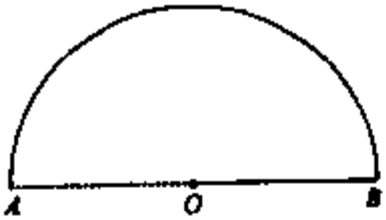
(iv) In the given diagram find $m\overline{OX}$ if $m\overline{OA}$ = 6cm and $m\overline{OY}$ = 9cm

(a) 4cm (b) 6cm
(c) 9cm (d) 12cm



(v) In the adjacent figure find semicircular area if π = 3.1416 and $m\overline{OA}$ = 20cm.

(a) 62.83 sq cm (b) 314.16 sq cm
(c) 436.20 sq cm (d) 628.32 sq cm



(vi) In the adjacent figure find half the perimeter of circle with centre O if $\pi = 3.1416$ and $m\overline{OA} = 20$cm.
(a) 31.42 cm (b) 62.832 cm
(c) 125.65 cm (d) 188.50 cm

(vii) A line which has two points in common with a circle is called.
(a) sine of a circle (b) cosine of circle
(c) tangent of a circle (d) secant of a circle

(viii) A line which has only one point in common with a circle is called
(a) sine of a circle (b) cosine of circle
(c) tangent of a circle (d) secant of a circle

(ix) Two tangents drawn to a circle from a point outside it are of .... In length.
(a) half (b) equal
(c) double (d) triple

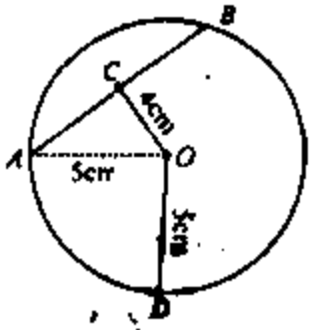
(x) A circle has only one
(a) secant (b) chord
(c) diameter (d) centre

(xi) A tangent line intersects the circle at
(a) three points (b) two points
(c) single points (d) no point at all

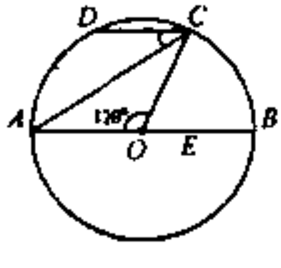
(xii) Tangents drawn at the ends of diameter of a circle are ..... to each other.
(a) parallel (b) non parallel
(c) collinear (d) perpendicular

(xiii) The distance between the centres of two congruent touching circles externally is
(a) of zero length (b) the radius of each circle
(c) the diameter of each circle
(d) twice the diameter of each circle

(xiv) In the adjacent circular figure with centre O and radius 5cm. The length of the chord intercepted at 4cm away from the centre of this circle is
(a) 4cm (b) 6cm
(c) 7cm (d) 9cm



(xv) In the adjoining figure there is a circle with centre O. If $\overline{DC} \parallel$ diameter $\overline{AB}$ and $m\angle AOB = 120^{\circ}$, then $m\angle ACD$ is
(a) 40° (b) 30°
(c) 50° (d) 60°



Answers:

(i)	c	(ii)	a	(iii)	b	(iv)	a	(v)	d	(vi)	b
(vii)	d	(viii)	c	(ix)	b	(x)	d	(xi)	c	(xii)	a
(xiii)	c	(xiv)	b	(xv)	b						

11 CHORDS AND ARCS

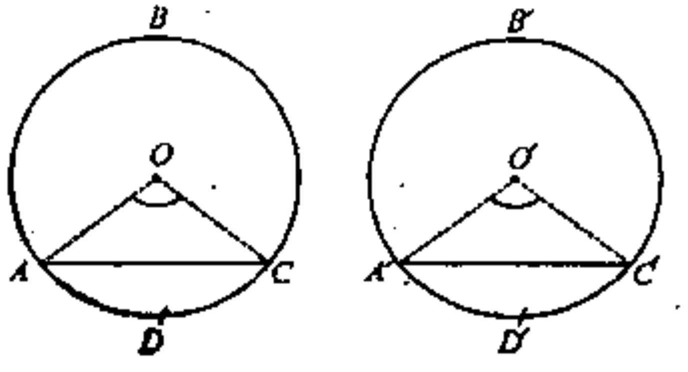
In this unit, students will learn how to

Prove the following theorems along with corollaries and apply them to solve appropriate problems.

- If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.
- If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent.
- Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
- If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.

THEOREM 1

If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.



Given: $ABCD$ and $A'B'C'D'$ are two congruent circles with centres O and O' respectively. So that $m\,\widehat{ADC} = m\,\widehat{A'D'C'}$

To prove:

$m\overline{Ac} = \overline{A'c'}$

Construction:

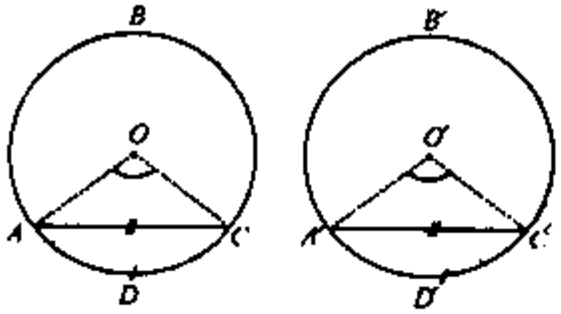
Join O with A, O with C, O' with A' and O' with C'. So that we can form $\Delta'\,OAC$ and $O'A'C'$.

Proof:

Statements	Reasons
In two equal circles $ABCD$ and $A'B'C'D'$ with centres O and O' respectively.	Given
$M\,\widehat{ADC} = m\,\widehat{A'D'C'}$	Given
$\therefore\ m\angle AOC = m\angle A'O'C'$	Central angles subtended by equal arcs of the equal circles.
Now in $\Delta AOC \leftrightarrow \Delta A'O'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of equal circles
$m\angle AOC = m\angle A'O'C'$	Already Proved
$m\overline{OC} = m\overline{O'C'}$	Radii of equal circles
$\therefore\ \Delta AOC \cong \Delta A'O'C'$	S.A.S ≅ S.A.S
and in particular $m\overline{AC} = m\overline{A'C'}$	

THEOREM 2

If two chords of a circle (or of congruent circles) are equal, then their corresponding are (minor, major or semi-circular) are congruent. In equal circles or in the same circle, if two chords are equal, they cut off equal arcs.



Given: $ABCD$ and $A'B'C'D'$ are two congruent circles with centres O and O' respectively. So that $m\overline{AC} = \overline{A'C'}$

To prove:

$m\,\widehat{ADC} = m\,\widehat{A'D'C'}$

Construction:

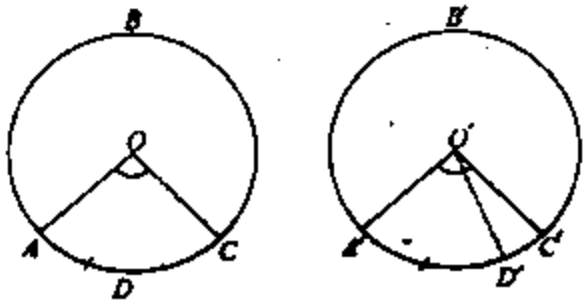
Join O with A, O with C, O' with A' and O' with C'.

Proof:

Statements	Reasons
In $\Delta AOC \leftrightarrow \Delta A'O'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of equal circles
$m\overline{OC} = m\overline{O'C'}$	Radii of equal circles
$m\overline{AC} = m\overline{A'C'}$	Given
$\therefore$ $\Delta AOC \cong \Delta A'O'C'$	S.S.S $\cong$ S.S.S
$\Rightarrow$ $m\angle AOC = m\angle A'O'C'$	
Hence $m\,\widehat{ADC} = m\,\widehat{A'D'C'}$	Arcs corresponding to equal central angles.

THEOREM 3

Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).



Given: $ABCD$ and $A'B'C'D'$ are two congruent circles with centres O and O' respectively. So that $m\overline{AC} = m\overline{A'C'}$

To prove:

$\angle AOC \cong \angle A'O'C'$

Construction:

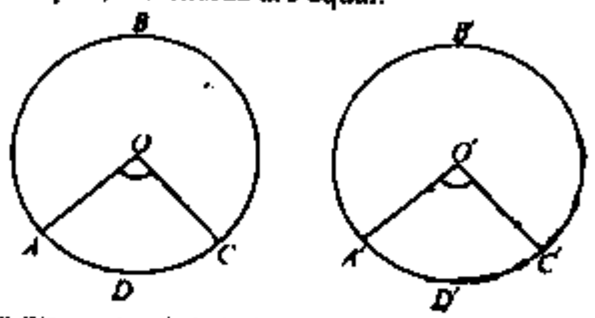
Let if possible $m\angle AOC \neq m\angle A'O'C'$ then consider $\angle AOC \cong \angle A'O'D'$

Proof:

Statements	***Reasons***
$\angle AOC \cong \angle A'O'D'$	Construction
$\therefore$ $\widehat{AC} \cong \widehat{A'C'}$ (i)	Arcs subtended by equal Central angles in congruent circles
$\overline{AC} = \overline{A'D'}$ (ii)	Using Theorem I
But $\overline{AC} = \overline{A'C'}$ (iii)	Given
$\therefore$ $\overline{A'C'} = \overline{A'D'}$	Using (ii) and (iii)
Which is only possible, if C' coincides with D'.	
Hence $m\angle A'O'C' \cong m\angle A'O'D'$ (iv)	
But $\angle AOC \cong \angle A'O'D'$ (v)	Construction
$\Rightarrow$ $\angle AOC \cong \angle A'O'C'$	Using (iv) and (v)

THEOREM 4

If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.



Given: $ABCD$ and $A'B'C'D'$ are two congruent circles with centres O and O' respectively.

So that $m\angle AOC \cong m\angle A'O'C'$

To prove:

$m\overline{AC} = m\overline{A'C'}$

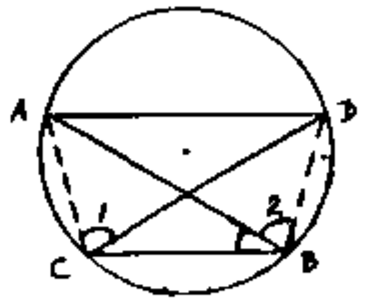
Proof:

Statements	Reasons
Since ABCD and $A'B'C'D'$ are two congruent circles with centres O and O' respectively.	Given
Place the circle ABCD on the circle $A'B'C'D'$ so that point O falls on O'. (i)	
Also $m\angle AOC = m\angle A'O'C'$	Given
$\therefore$ $m\overline{OA} = m\overline{O'A'}$ (ii)	Radii for congruent circles
and $m\overline{OC} = m\overline{O'C'}$ (iii)	Radii for congruent circles
So point A will coincide with A' and point C will coincide with C'.	Using (i), (ii) and (iii)
Thus, $m\overline{AC} = m\overline{A'C'}$	

EXERCISE 11.1

1. ***In a circle two equal chords $\overline{AB}$ and $\overline{CD}$ intersect each other.***

Prove that $m\overline{AC} = m\overline{BD}$.



Solution: In the given figure $m\overline{AB} = m\overline{CD}$

To prove:

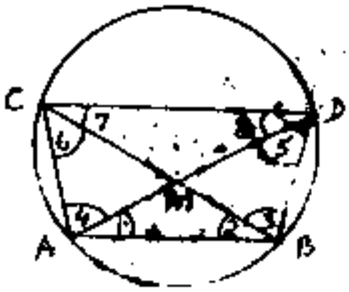
$m\overline{AC} = m\overline{BD}$

Proof:

Statements	***Reasons***
In Δ^s CBD and BCA	
$m\overline{CB} = m\overline{CB}$	Common
Now $m\overline{AB} = m\overline{CD}$	Given
and $m\angle 2 = m\angle 1$ (i)	$\overline{DA}$ is chord, $\angle 1$, $\angle 2$ in the same segement.
and $m\angle CBD = m\angle BCA$ (ii)	$m\overline{AB} = m\overline{CD}$ their opposite angles in the same segment.
$m\angle CBD - m\angle 2 = m\angle BCA - m\angle 1$	from (ii) – (i)

$m\angle CBA \quad m\angle BCD$	
Now $\overline{CB}$, $\overline{BA}$ and their included $\angle CBA$ is congruent to $\overline{CB}$, $\overline{CD}$ and their included $\angle BCD$	
Thus $\Delta CBD \cong \Delta BCA$	
Thus $\overline{AC} \quad m\overline{BD}$ proved.	

2. ***In a circle prove that the arcs between two parallel chords are equal.***



Given: In the given figure $\overline{AB} \parallel \overline{CD}$

To prove:

Arc $m\overline{AC}$ Arc $m\overline{BD}$

Construction:

Join A with D and B with C.

Proof:

Statements	*Reasons*
In the figure	
$m\angle 1 \quad m\angle 8$ (i) (Alt. angles)	$\overline{AB} \parallel \overline{CD}$
$m\angle 2 \quad m\angle 7$ (ii) (Alt. angles)	$\overline{AB} \parallel \overline{CD}$
$m\angle 1 \quad m\angle 7$ (iii) angles in the same segment on $\overline{BD}$ chord.	
Therefore, $m\angle 7 \quad m\angle 8$ (i), (ii)	

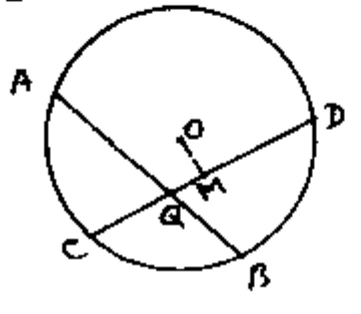
Thus, in ΔCMD	
$m\angle DM = m\angle CM$ (iv)	
Similarly $m\overline{MA} = m\overline{MB}$ (v)	
Thus $m\overline{DM} + m\overline{MA} = m\overline{CM} + m\overline{MB}$	from (iv) + (v)
or $m\overline{AD} = m\overline{BC}$ (vi)	
Now in Δ^s ABD and ABC	
$m\overline{AB} = m\overline{AB}$	Common
$m\angle 1 = m\angle 2$	from (ii), (iii)
$m\overline{AD} = m\overline{BC}$	From (vi)
Thus, $\Delta ABD \cong \Delta ABC$	
$\therefore$ $m\overline{BD} = m\overline{AC}$	
Hence $\widehat{BD} = \widehat{AC}$	Proved.

3. ***Give a geometric proof that a pair of bisecting chords are the diameters of a circle.***

Given: In the figure, let $\overline{AB}$ and $\overline{CD}$ chords bisect each other point Q, then

$m\overline{CQ} = m\overline{QD}$ and

$m\overline{BQ} = m\overline{QA}$



To prove:

$\overline{AB}$ and $\overline{CD}$ pass through point O, centre of the circle.

Construction:

Draw $\overline{OM} \perp \overline{CD}$.

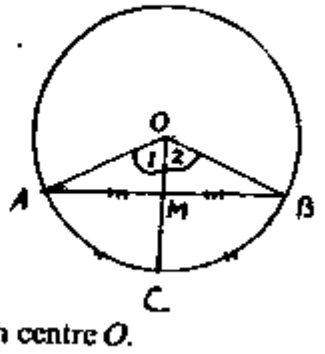
Proof:

Statements	Reasons
$\overline{OM} \perp \overline{CD}$	Const.
$\therefore$ $m\overline{CM} = m\overline{DM}$	Theorem
Thus, M is the mid point of $\overline{CD}$ But Q is also the mid-point as supposed in given	Theorem

This is possible only when point Q, M coincide. Thus $\overline{OM}$ cannot be drawn perpendicular to $\overline{CD}$, point M must lie on point 'O' the centre of the circle.

Thus $\overline{CD}$ is the **diameter** of the circle. Similarly it can be proved that $\overline{AB}$ is the diameter of the circle.

4. ***If C is the mid point of an arc ACB in a circle with centre O. Show that line OC bisects the chord AB.***



Given: A circle with centre O.

$\widehat{ACB}$ is an arc and $m\,\widehat{AC} = m\,\widehat{CB}$

O is joined with C that $\overline{OC}$ cuts $\overline{AB}$ at M.

To prove:

$m\overline{AM} = m\overline{BM}$

Construction:

Join O with A and B.

Proof:

Statements	***Reasons***
$m\,\widehat{AC} = m\,\widehat{BC}$	Given
$\therefore \quad m\angle 1 = m\angle 2$	Theorem
Now in Δ^s AOM, BOM	
$m\overline{OM} = m\overline{OM}$	Common
$m\angle 1 = m\angle 2$	Proved
$m\overline{OA} = m\overline{OB}$	Radii of the same circle
$\Delta AOM \cong \Delta BOM$	
Hence $m\overline{AM} = m\overline{BM}$	Proved.

MISCELLANEOUS EXERCISE – 11

1. ***Multiple Choice Questions***

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) A 4cm long chord subtends a central angle of 60^O. The radial segment of this circle is:

(a) 1 (b) 2
(c) 3 (d) 4

(ii) The length of a chord and the radial segment of a circle are congruent, the central angle made by the chord will be:

(a) 30^O (b) 45^O
(c) 60^O (d) 75^O

(iii) Out of two congruent arcs of a circle, if one arc makes a central angle of 30^O then the other arc will subtend the central angle of:

(c) 45^O (d) 60^O

(iv) An arc subtends a central angle of 40^O then the corresponding chord will subtend a central angle of:
(a) 20^O (b) 40^O
(c) 60^O (d) 80^O

(v) A pair of chords of a circle subtending two congruent central angles is:
(a) congruent (b) incongruent
(c) over lapping (d) parallel

(vi) If an arc of a circle subtends a central angle of 60^O, then the corresponding chord of the arc will make the central angle of:
(a) 20^O (b) 40^O
(c) 60^O (d) 80^O

(vii) The semi circumference and the diameter of a circle both subtend a central angle of:
(a) 90^O (b) 180^O
(c) 270^O (d) 360^O

(viii) The chord length of a circle subtending a central angle of 180^O is always:
(a) less than radial segment
(b) equal to the radial segment
(c) double of the radial segment
(d) none of these

(ix) If a chord of a circle subtends a central angle of 60^O, then the length of the chord and the radial segment are:
(a) congruent (b) incongruent
(c) parallel (d) perpendicular

(x) The arcs opposite to incongruent central angles of a circle are always:
(a) congruent (b) incongruent
(c) parallel (d) perpendicular

Answers:

(i)	d	(ii)	c	(iii)	b	(iv)	b	(v)	a
(vi)	c	(vii)	b	(viii)	c	(ix)	a	(x)	b

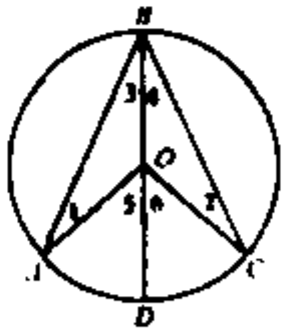
SUMMARY

- The boundary traced by a moving point in a circle is called its circumference whereas any portion of the circumference will be known as an arc of the circle.
- The straight line joining any two points of the circumference is called a chord of the circle.
- The portion of a circle bounded by an arc and a chord is known as the segment of a circle.
- The circular region bounded by an arc of a circle and its two corresponding radial segments is called a sector of the circle.
- A straight line, drawn from the centre of a circle bisecting a chord is perpendicular to the chord and conversely perpendicular drawn from the centre of a circle on a chord, bisects it.
- If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.
- If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent.
- Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
- If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.

12 ANGLE IN A SEGMENT OF A CIRCLE

THEOREM 1

The measure of a central angle of a minor arc of a circle, is double that the angle subtended by the corresponding major arc.



Given:	$\widehat{AC}$ is an arc of a circle with centre O.
Whereas	$\angle AOC$ is the central angle and $\angle ABC$ is circum angle.
To prove:	$m\angle AOC \quad 2m\angle ABC$
Construction:	Join B with O and produce it to meet the circle at D.

Proof:

Statements		***Reasons***
As $\quad m\angle 1 \quad m\angle 3$	(i)	Angles opposite to equal sides in ΔOAB
and $\quad m\angle 2 \quad m\angle 4$	(ii)	Angles opposite to equal sides in ΔOBC
Now $\quad m\angle 5 \quad m\angle 1 + m\angle 3$	(iii)	External angle is the sum of internal opposite angles.
Similarly $m\angle 6 \quad m\angle 2 + m\angle 4$	(iv)	
Again $m\angle 5 \quad m\angle 3 + m\angle 3$		

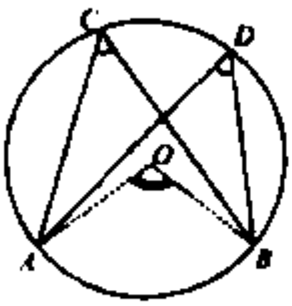
$2m\angle 3$	(v)	Using (i) and (iii)
and $m\angle 6 \quad m\angle 4 + m\angle 4$		
$2m\angle 4$	(vi)	Using (ii) and (iv)

Then from figure

$\Rightarrow \quad m\angle 5 + m\angle 6 \quad 2m\angle 3 + 2m\angle 4$	Adding (v) and (vi)
$\Rightarrow \quad m\angle AOC \quad 2(m\angle 3 \quad m\angle 4)$	
$2m\angle ABC$	

THEOREM 2

Any two angles in the same segment of a circle are equal.



Given: $\angle ACB$ and $\angle ADB$ are the circum angles in the same segment of a circle with centre O.

To prove: $m\angle ACB \quad m\angle ADB$

Construction: Join O with A and O with B.
So that $\angle AOB$ is the central angle.

Proof:

Statements	***Reasons***
Standing on the same arc AB of a circle.	
$\angle AOB$ is the central angle whereas	Construction
$\angle ACB$ and $\angle ADB$ are circum angles	Given
$\therefore \quad m\angle AOB \quad 2m\angle ACB \quad$ (i)	By theorem 1
and $\quad m\angle AOB \quad 2m\angle ADB \quad$ (ii)	By theorem 1
$\Rightarrow \quad 2m\angle ACB \quad 2m\angle ADB$	Using (i) and (ii)
Hence, $m\angle ACB \quad m\angle ADB$	

THEOREM 3

The angle

- in a semi-circle is a right angle,
- in a segment greater than a semi circle is less than a right angle,
- in a segment less than a semi-circle is greater than a right angle.

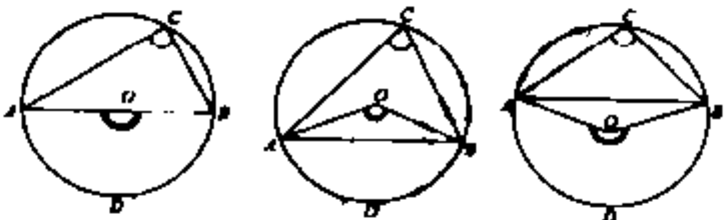


Fig. I Fig. II Fig. III

Given: $\overline{AB}$ is the chord corresponding to an arc ADB

Whereas $\angle AOB$ is a central angle and $\angle ACB$ is a circum angle of a circle with centre O.

To prove: In fig (i) If sector $\angle ACB$ is a semi circle then $m\angle ACB = 1\ \angle rt$

In fig (ii) If sector $\angle ACB$ is greater than a semi circle than $m\angle ACB < 1\ \angle rt$

In fig (iii) If sector ACB is less than a semi circle then $m\angle ACB > 1\ \angle rt$

Proof:

Statements	*Reasons*
In each figure, $\overline{AB}$ is the chord of a circle with center O.	Given
$\angle ACB$ is the circum angle	Given By theorem 1

Now in fig(I) $m\angle AOB$ 180° | A straight angle

$\therefore$ $m\angle AOB = 2\angle rt$ (ii)

$\Rightarrow$ $m\angle ACB = 1\angle rt$ | Using (i) and (ii)

In fig (II) $m\angle AOB < 180°$

$mACB\angle\frac{180}{2}$

$mACB\angle 90$

In fig. (III)

$mACB\angle\frac{180}{2}$

$mACB\angle 90$

THEOM 4

The opposite angles ofuadrilateral inscribed in a circle are supplementa

Given: ABCD is a quadrilateral in a circle with centre O.

To prove: $m\angle A + m\angle C$

$m\angle B + m\angle D$

Construction: Draw $\overline{OA}$ anc

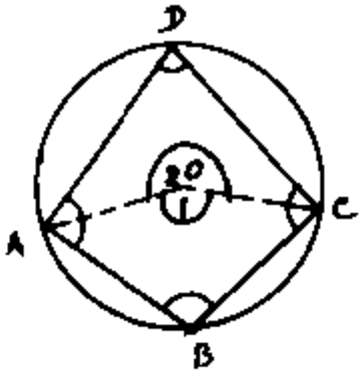
Proof:

Statements	*Reasons*
Standing on the same arc AI central angle.	Arc ADC of the circle with centre O.
Whereas $\angle B$ is the circum an	

$\therefore \quad m\angle B = \frac{1}{2}(m\angle 2)$ (i)	By theorem 1
Standing on the same arc ABC4 is a central angle whereas $\angle D$ the circum angle.	Arc ABC of the circle with centre O.
$\therefore \quad m\angle D = \frac{1}{2}(m\angle 4)$ (ii)	By theorem 1
$\Rightarrow \quad m\angle B + m\angle D = \frac{1}{2}m\angle 2 + \angle 4$	Adding (i) and (ii)
$= \frac{1}{2}(m\angle 2 + m\angle 4)$	
$= \frac{1}{2}$(Total central angle)	
i.e., $m\angle B + m\angle D = \frac{1}{2}(4\angle rt)$	
Similarly $m\angle A + m\angle C = 2\angle rt$	

EXERCISE 12.1

Q.1 Prove that in a given cyclic quadrilateral, sum of opposite angles is two right angles and conversely.



Given: A circle with center at O.
ABCD is a cyclic quadrilateral.

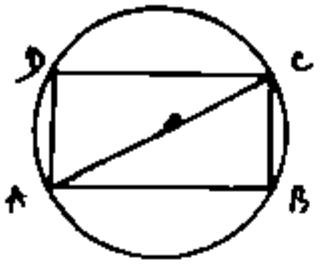
To prove: $m\angle B + m\angle D = 180°$
$m\angle BCD + m\angle DAB = 180°$

Construction: Join O with A and C

Proof:

Statements	Reasons
$m\angle 1 = 2m\angle D$ (i) $m\angle 2 = 2m\angle B$ (ii)	Angles 1, 2 are angles at the centre and $\angle D$, $\angle B$ are angles in arcs.
$m\angle 1 + m\angle 2 = 2m\angle D + 2m\angle B$	Adding (i), (ii)
or $2[m\angle D + m\angle B] = m\angle 1 + m\angle 2$ $= 360°$	Angles at a point
$m\angle D + m\angle B = \dfrac{360}{2}$	Dividing by 2
$m\angle D + m\angle B = 180°$	Proved
Similarly $m\angle BCD + m\angle DAB = 180°$	

Q.2 Show that parallelogram inscribed in a circle will be a rectangle.



Given: ABCD is a parallelogram inscribed in the circle with centre O.

$m\overline{AB}$ $m\overline{DC}$ and $\overline{AB} \parallel \overline{DC}$

$m\overline{AD}$ $m\overline{BC}$ and $\overline{AD} \parallel \overline{BC}$

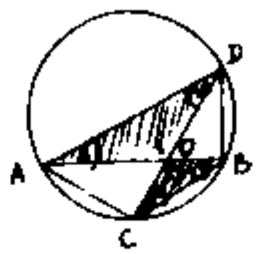
To prove: ABCD is a rectangle

Construction: Join A with C.

Proof:

Statements	***Reasons***
In Δ' ABC and ADC	
$m\overline{AC}$ $m\overline{AC}$	Common
$m\overline{AB}$ $m\overline{DC}$	Given
$m\overline{BC}$ $m\overline{AD}$	Given
$\therefore$ $\Delta ABC \cong \Delta ADC$	S.S.S. $\cong$ S.S.S
Thus, $m\angle B$ $m\angle D$ (i)	
But $m\angle B + m\angle D$ 180° (ii)	Opposite angles of cyclic quadrilateral
From (i) and (ii)	
$m\angle B$ $m\angle D$ 90°	
Similarly $m\angle BAD$ $m\angle BCD$ 90°	
Hence, ABCD is a rectangle	

Q.3 AOB and COD are two intersecting chords of a circle. Show that Δ' AOD and BOC are equiangular



Given: In a circle AOB and BOC are two intersecting chords at joint O.

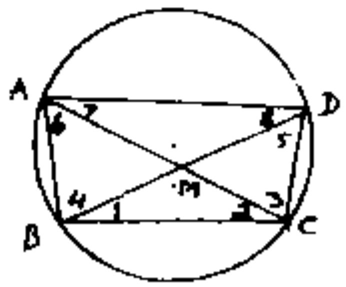
To prove: Δ^s AOD and BOC are equiangular.

Construction: Join A with C and C with B.

Proof:

Statements		*Reasons*
$\angle 1 \cong \angle 2$	(i)	Vertical opposite angles.
AC is chord and angle 3, 4 are in the same segment.		
Therefore, $\angle 3 \cong \angle 4$	(ii)	
Now BD is chord and angle 5, 6 are in the same segment.		
Therefore, $\angle 5 \cong \angle 6$	(iii)	
Thus, Δ AOB and COD are equiangular.		From i, ii, iii

Q.4 $\overline{AD}$ ***and*** $\overline{BC}$ ***are two parallel chords of a circle, prove that arc*** $AB \cong$ ***arc*** CD ***and arc*** $AC \cong$ ***arc*** BD



Given: In a given circle $AD \parallel BC$

To prove: arc $AC \cong$ arc BD

Construction: Join A with C and B with D AC and BD intersect a M.

Proof:

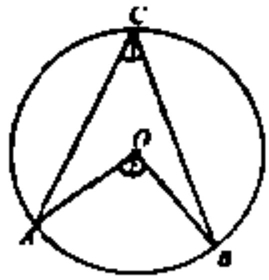
Statements	***Reasons***
In this figure	
$m\angle 1$ $m\angle 8$ (i) $AD \parallel BC$	Alt. angles
$m\angle 2$ $m\angle 7$ (ii) $AD \parallel BC$	Alt angles
$m\angle 1$ $m\angle 7$ (iii)	Angles in the same segment.
Therefore $m\angle 7$ $m\angle 8$	From (i),(iii)
Thus, in ΔAMD	
$m\,AM$ $m\,DM$ (iv)	
Similarly $m\,MC$ $m\,BM$ (v)	
Thus, $m\,AM + m\,MC$ $m\,\overline{DM} + m\,\overline{BM}$	From (iv)+(v)
or $m\,AC$ $m\,BD$ (iv)	
Now in Δ BCA and BCD	
$m\,BC$ $m\,\overline{BC}$	Common.
$m\angle 1$ $m\angle 2$	From (ii) (iii)
$m\,AC$ $m\,\overline{BD}$	From iv
Thus $\Delta BCD \cong \Delta BCA$	
$\therefore$ $AB \cong CD$	Proved
Hence, $AB \cong CD$	

MISCELLANEOUS EXERCISE – 12

Q.1 ***Multiple Choice Questions.***
Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) A circle passes through the vertices of a right angled ΔABC with $m\overline{AC}$ 3cm and $m\overline{BC}$ 4cm, $m\angle C$ 90°. Radius of the circle is:
(a) 1.5cm (b) 2.0cm (c) 2.5cm (d) 3.5cm

(ii) In the adjacent circular figure, central and inscribed angles stand on the same arc AB, then

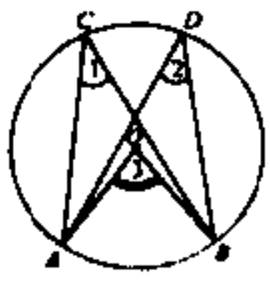


(a) $m\angle 1 \quad m\angle 2$ (b) $m\angle 1 \quad 2\angle m2$
(c) $m\angle 2 \quad m\angle 3$ (d) $m\angle 2 = 2\angle m1$

(iii) In the adjacent figure if $m\angle 3$ 75°, then find $m\angle 1$ and $m\angle 2$.

(a) $37\frac{1}{2}$, $37\frac{1}{2}$ (b) $37\frac{1}{2}$, 75°

(c) 75°, $37\frac{1}{2}$ (d) 75°, 75°



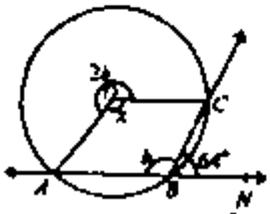
(iv) Given that O is the centre of the circle. The angle marked x will be:

(a) $12\frac{1}{2}$ (b) 25° (c) 50° (d) 75°

(v) Given that O is the centre of the circle the angle marked y will be:

(a) $12\frac{1}{2}$ (b) 25° (c) 50° (d) 75°

(vi) In the figure, O is the centre of the circle and ABN is a straight line. The obtuse angle AOC x is:



(a) 32° (b) 64° (c) 96° (d) 128°

(vii) In the figure, O is the centre of the circle, then the angle x is:

(a) 55° (b) 110° (c) 220° (d) 125°

(viii) In the figure, O is the centre of the circle then angle x is:

(a) 15° (b) 30° (c) 45° (d) 60°

(ix) In the figure, O is the centre of the circle then the angle x is:

(a) 15° (b) 30° (c) 45° (d) 60°

(x) In the figure, O is the centre of the circle then the angle x is:

(a) 50° (b) 75° (c) 100° (d) 125°

Answers:

(i)	C	(ii)	d	(iii)	a	(iv)	c	(v)	b
(vi)	d	(vii)	d	(viii)	b	(ix)	d	(x)	c

13 PRACTICAL GEOMETRY-CIRCLES

After studying this unit, students will learn how to

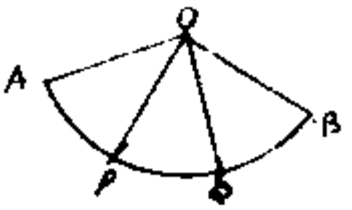
- locate the centre of a given circle.
- draw a circle passing through three given non-collinear points.
- complete the circle when a part of its circumference is given,
  - (i) by finding the centre,
  - (ii) without finding the centre.
- circumscribe a circle about a given triangle.
- inscribe a circle in a given triangle.
- escribe a circle in a given triangle.
- circumscribe an equilateral triangle about a given circle.
- inscribe an equilateral triangle in a given circle.
- circumscribe a square about a given circle.
- inscribe a square in a given circle.
- circumscribe a regular hexagon about a given circle.
- inscribe a regular hexagon in a given circle.
- draw a tangent to a given arc, without using the centre, through a given point p when p is the middle point of the arc, p is at the end of the arc and p is outside the arc.
- draw a tangent to a given circle from a point P when P is on the circumference and when p is outside the circle.
- draw two tangents to a circle meeting each other at a given angle
- draw direct common tangent or external tangents to two equal circles and draw transverse common tangents or internal tangents to two equal circles.
- draw direct common tangents or external tangents to two unequal circles and draw transverse common tangents or internal tangents to two unequal circles.
- draw a tangent to two unequal touching circles and two unequal intersecting circles
- draw a circle which touches
  - (i) both the arms of a given angle.
  - (ii) two converging lines and passes through a given point between them.
  - (iii) three converging lines.

EXERCISE 13.1

1. Divide an arc of any length

(i) into three equal parts.

By hit and trial method, we find point P, Q that $m\,\widehat{AP}$ $m\,\widehat{PQ}$ $m\,\widehat{QB}$



(ii) into four equal parts.

Construction:

$\widehat{AQB}$ is the given arc.

Steps: Point Q is mid point of the arc.

(ii) Join Q with A, B

(iii) Find mid-points of $\widehat{APQ}$ arc and $\widehat{QRB}$ arc.

The given arc has been divided at points P Q R into four equal parts.

(iii) into six equal parts.

(i) Divide $\widehat{AB}$ into 3 equal parts at C, D points.

(ii) Bisect arcs $\widehat{AC}$, $\widehat{CD}$ and $\widehat{DB}$ at points E, F and G respectively.

Arc $\widehat{AB}$ has been divided to six equal parts as:

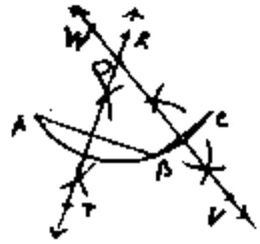
$\widehat{AE}$, $\widehat{EC}$, $\widehat{CF}$, $\widehat{FD}$, $\widehat{DG}$ and $\widehat{GB}$.

2. *Practically find the centre of an arc ABC.*

Steps of Construction:

(i) Draw $\overline{AB}$ and $\overline{BC}$ chords of the given arc.

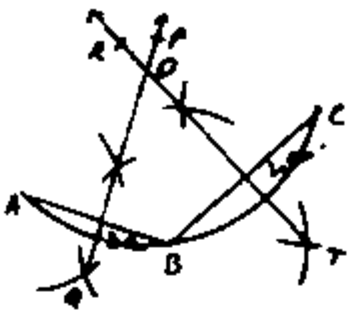
(ii) Draw right bisectors $\overleftrightarrow{RT}$ and $\overleftrightarrow{WV}$ of $\overline{AB}$ and $\overline{BC}$.
These right bisectors intersect each other at O.



Result: O is the required centre of the arc.

3. (i) If $|\overline{AB}| = 3\text{cm}$ and $|\overline{BC}| = 4\text{cm}$ are the lengths of two chords of an arc, then locate the centre of the arc.

Given: $m\overline{AB} = 3\text{cm}$, $m\overline{BC} = 4\text{cm}$ are the chord of $\widehat{ABC}$.



Required:

To find the centre of $\widehat{ABC}$.

Construction:

(i) Draw $\overleftrightarrow{PQ}$ right bisector of $\widehat{AB}$

(ii) Draw $\overleftrightarrow{RT}$ right bisector of $\widehat{BC}$

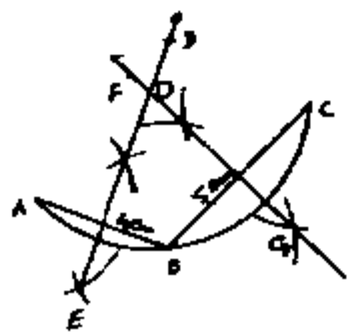
These right bisectors intersect each other at point O.

Result:

O is the centre of the arc $\widehat{ABC}$.

(ii) If $|\overline{AB}|$ = 3.5cm and $|\overline{BC}|$ = 5cm are the lengths of two chords of an arc, then locate the centre of the arc.

Given: $\widehat{ABC}$ is an arc in which $m\overline{AB}$ = 4 cm and measure chord $\overline{BC}$ 5 cm.



Required:

To locate the centre of the arc $\widehat{ABC}$.

Construction:

(i) Draw right bisector $\overleftrightarrow{DE}$ of $\overline{AB}$

(ii) Draw right bisector $\overleftrightarrow{DE}$ of $\overline{BC}$.

$\overleftrightarrow{DE}$ and $\overleftrightarrow{FG}$ intersect at point O.

Result:

O is the required centre of the arc $\widehat{ABC}$.

4. ***For an arc draw two perpendicular bisectors of the chords $\overline{PQ}$ and $\overline{QR}$ of this arc, construct a circle through P, Q and R.***

Construction:

$\widehat{PQR}$ is an arc. $\overline{PQ}$ and $\overline{QR}$ are its two chords.

(i) $\overleftrightarrow{BA}$ is right bisector of $\overline{PQ}$.

(ii) $\overleftrightarrow{CD}$ is right bisector of $\overline{QR}$.

$\overline{BA}$ and $\overline{CD}$ intersect at point O.

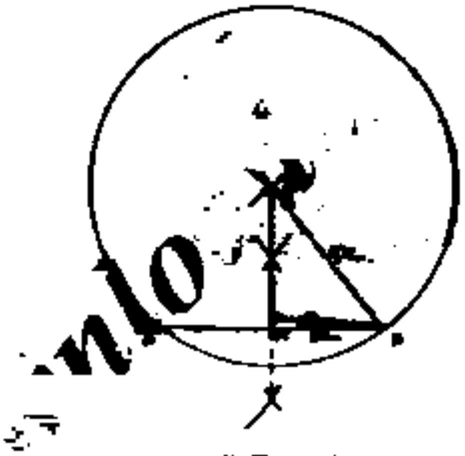
(iii) Take O as centre and draw a circle of radius $m\overline{OP}$ or $m\overline{OQ}$ or $m\overline{OR}$, this passes through P, Q and R.

5. ***Describe a circle of radius 5cm passing through points A and B, 6cm apart. Also find distance from the centre to the line segment AB.***

(i) A, B are two points that $m\overline{AB} = 6$cm.

(ii) Take A and B as centres and draw two arcs of radius 5 cm that intersect at point O.

(iii) Take O as centre and draw a circle of $m\overline{AB} = 5$cm that passes through A and B.



(iv) Drop perpendicular $\overline{OM}$ on $\overline{AB}$. $\overline{OM}$ is distance of O from $\overline{AB}$.

Calculations:

In ΔOMB

$m\overline{OB}$ 5 cm

$m\overline{MB}$ $\frac{6}{2}$ 3 cm

By Pythagorean Theorem

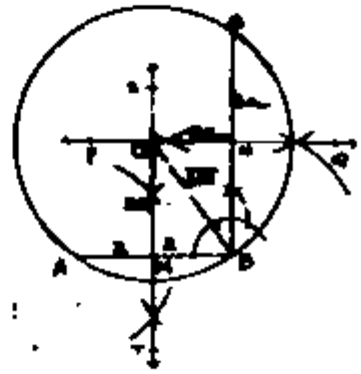
$$(m\overline{OM})^2 = (m\overline{OB})^2 - (m\overline{MB})^2$$

$$(m\overline{OM})^2 = (5)^2 - (3)^2$$
$$= 25 - 9$$
$$= 16$$

$$m\overline{OM} = \sqrt{16} = 4 \text{ cm}$$

6. ***If $|\overline{AB}| = 4cm$ and $|\overline{BC}| = 6cm$, such that $\overline{AB}$ is perpendicular to $\overline{BC}$, construct a circle through points A, B and C. Also measure its radius.***

Steps of Construct



(i) Draw $\overline{AB}$ 4 cm long.

(ii) Take $\overline{BC} \perp \overline{AB}$ at B and cut off $m\overline{BC} = 6$ cm.

(iii) Draw $\overline{ST} \perp \overline{AB}$ and $\overline{PQ} \perp \overline{BC}$, $\overline{ST}$ and $\overline{PQ}$ point O.

(iv) Take O as centre and draw a circle with radius $\overline{OB}$.

This passes through A, B, C. $\overline{OB}$ is radius of the circle.
Now, in right angled triangle OMB.

$$m\overline{OB} = \sqrt{(m\overline{OM})^2 + (m\overline{MB})^2}$$
$$= \sqrt{(3)^2 + (2)^2}$$

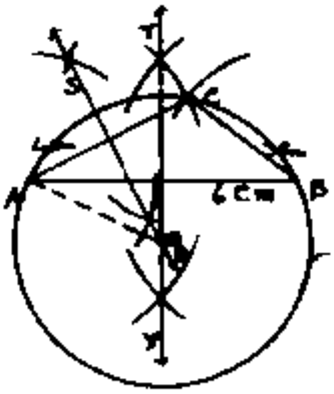
$\sqrt{9+4}$
$\sqrt{13}$ cm.

EXERCISE 13.2

1. Circumscribe a circle about a triangle ABC with sides

$|\overline{AB}| = 6cm$, $|\overline{BC}| = 3cm$, $|\overline{CA}| = 4cm$
Also measure its circum radius.

Steps of Construction



(i) Draw a triangle ABC having

$m\overline{AB}$ 6 cm

$m\overline{BC}$ – 3 cm and

$m\overline{CA}$ 4 cm .

(ii) Draw $\overline{TV}$ right bisector of $\overline{AB}$

(iii) Draw $\overline{SR}$ right bisector of $\overline{AC}$.

$\overline{TV}$ and $\overline{SR}$ intersect each other at point O. This is the centre of the required circle.

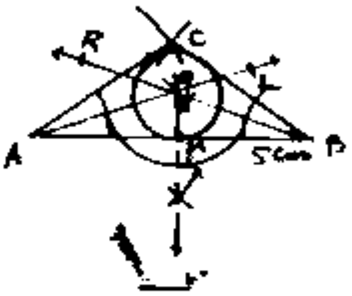
(iv) Take O as centre and draw a circle with radius $m\overline{OA}$. This circle passes through A, B, C.

$m\overline{OA}$ 3.3 cm.

2. ***Inscribe a circle in a triangle ABC with sides***

$|\overline{AB}| = 5cm$, $|\overline{BC}| = 3cm$, $|\overline{CA}| = 3cm$. Also measure its in-radius.

Steps of Construction:



(i) Draw a ΔABC with $m\overline{AB} = 5cm$

$m\overline{BC}$ 3cm

$m\overline{CA}$ 3cm

(ii) Draw $\overrightarrow{AL}$ bisector of angle $\angle A$

(iii) Draw $\overrightarrow{BR}$ bisector of angle $\angle B$

$\overrightarrow{AL}$ and $\overrightarrow{BR}$ intersect at O.

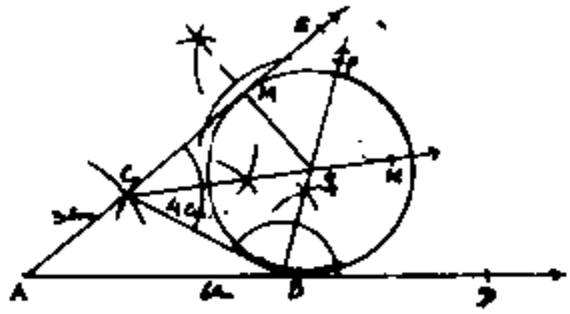
(iv) Draw $\overline{OM} \perp AB$.

(v) Take O as centre and draw a circle with $\overline{OM}$ as radius. This circle touches the three sides of the triangle.

$m\overline{OM}$ .8 cm

3. ***Describe a circle opposite to vertex A to a triangle ABC with sides $|\overline{AB}| = 6cm$, $|BC| = 4cm$ $|\overline{CA}| = 3cm$. Find its radius also.***

Steps of Construction:



(i) Draw a triangle ABC with

$m\overline{AB} = 6$ cm

$m\overline{BC} = 4$ cm

$m\overline{CA} = 3$ cm

(ii) Extend $\overline{AB}$ towards B and $\overline{AC}$ towards C.

(iii) Draw $\overrightarrow{BP}$ bisector of $\angle CBD$

(iv) Draw $\overrightarrow{CN}$ bisector of $\angle BCE$.

$\overrightarrow{BP}$ and $\overrightarrow{CN}$ intersect at point I,

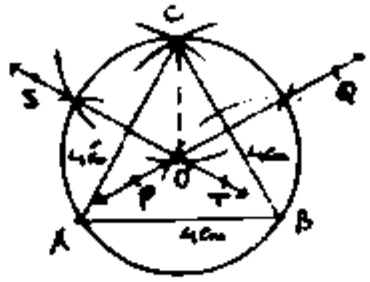
(v) Drop $\overline{IM} \perp \overline{CE}$

(vi) Take I, as centre and drawn the circle with $\overline{I_1M}$ as radius.

This circle touches $\overline{CB}$ externally and $\overline{AC}$ and $\overline{AB}$ internally.

$m\overline{IM} = 2.3$ cm.

4. ***Circumscribe a circle about an equilateral triangle ABC with each side of length 4cm.***



(i) Draw a triangle ABC with $m\overline{AB} = m\overline{BC} = m\overline{CA} = 4$ cm.

(ii) Draw $\overleftrightarrow{PQ}$ right bisector of $\overline{BC}$

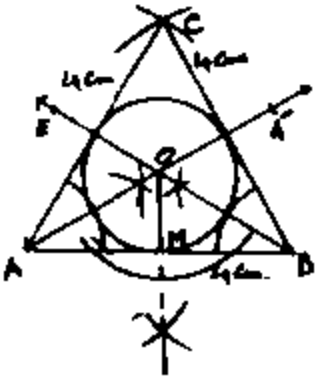
(iii) Draw $\overleftrightarrow{ST}$ right bisector of $\overline{AC}$

$\overleftrightarrow{PQ}$ and $\overleftrightarrow{ST}$ intersect at point O.

(iv) Take O as centre and draw a circle with radius $m\overline{OC}$.
This circle passes through vertices A, B and C.

5. ***Inscribe a circle in an equilateral triangle ABC with each side of length 5cm.***

Steps of Construction:



(i) Draw a ΔABC with each side = 5cm.

(ii) Draw $\overrightarrow{AA'}$ bisector of $\angle A$.

(iii) Draw $\overrightarrow{BE}$ bisector of $\angle B$.

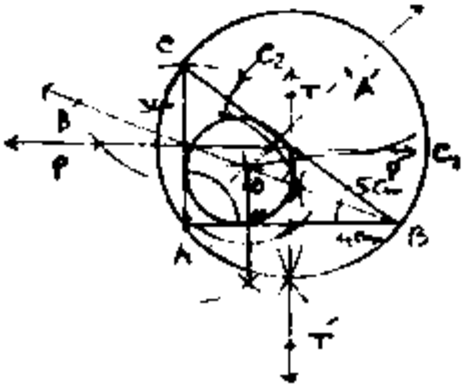
$\overrightarrow{AA'}$ and $\overrightarrow{BE}$ intersect at point O.

(iv) Drop $\overline{OM} \perp \overline{AB}$.

(v) Take O as centre and draw a circle with $m\overline{OM}$ as radius. This is inscribed circle to triangle ABC.

6. ***Circumscribe and inscribe with regard to a right angle triangle with sides, 3cm, 4cm and 5cm.***

Steps of Construction:



(i) Draw a ΔABC in which $m\overline{AB} = 4$ cm

$m\overline{AC} = 3$ cm

$m\overline{BC} = 5$ cm.

(ii) Draw $\overleftrightarrow{TT'}$ right bisector of $\overline{AB}$.

(iii) Draw $\overleftrightarrow{PP'}$ right bisector of $\overline{AC}$.

Right bisectors $\overleftrightarrow{TT'}$ and $\overleftrightarrow{PP'}$ intersect at O.

(iv) Take O as centre and draw a circle with radius $\overline{OB}$. This circle passes through vertices A, B and C. This circumscribe circle.

(v) Draw bisector of $\angle A$ as $\overrightarrow{AA'}$ and bisector of $\angle B$ as $\overrightarrow{BB'}$.

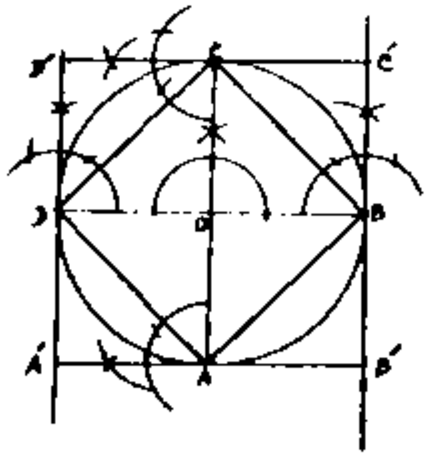
$\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ intersect at O.

(vi) Drop $\overline{OM} \perp \overline{AB}$.

(vii) Take O as centre and draw a circle of radius $m\overline{OM}$. This is inscribed circle and touches the sides of the triangle.

7. *In and about a circle of radius 4cm describe a square.*

Steps of Construction:



(i) Take any point O.

(ii) Take O as centre and draw a circle of 4 cm radius.

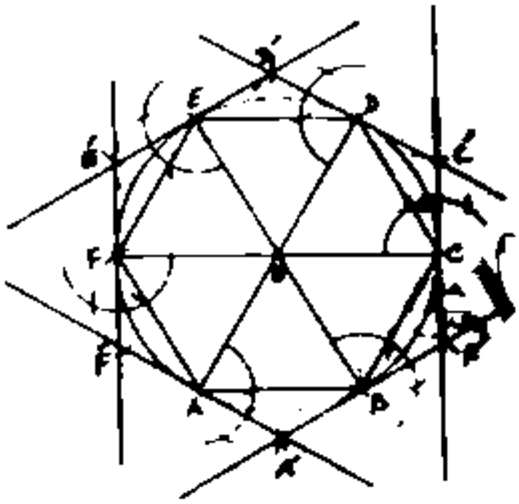
(iii) Draw $\overline{AC}$ and $\overline{DB}$ two diagonals perpendicular to each other.

(iv) Join A to B and D, join C to D and B.
$ABCD$ is the required square in the circle.

(v) Draw $B'C'$ tangent to the circle at point B.
Draw $C'D'$ tangent to the circle at point C.
Draw $D'A'$ tangent to the circle at point D.
Draw $A'B'$ tangent to the circle at point A.
$A'B'C'D'$ is the required square about the circle.

8. *In and about a circle of radius 3.5cm describe a regular hexagon.*

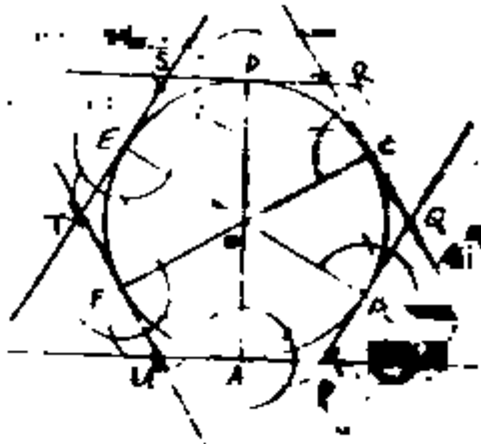
Steps of Construction:



(i) Take any point O.
(ii) Take O as centre and draw a circle of radius 3.5cm.
(iii) Take any point A on the circumference.
(iv) Take A as centre and mark a point B on the circumference that $m\overline{AB} = m\overline{OA}$,
Similarly mark point C, D, E, F on the circumference.
$ABCDEF$ is the required hexagon in the circle.
(v) Join O with A, B, C, D, EF.
(vi) Draw tangents to the circle at points A, B, C, D, E, F.
These tangents intersect at points A', B', C', D', E', F'.
A', B', C', D', E', F' is the required hexagon about the circle.

9. ***Circumscribe a regular hexagon about a circle of radius 3cm.***

Steps of Construction:



(i) Take a point O as centre and draw a circle of radius 3 cm.

(ii) Take any point A on the circumference.

(iii) Starting from A, mark points B, C, D, E, F at equal distance on the circumference, with the help of the length of the radius.

(iv) Join O with A, B, C, D, E, F.

(v) Draw tangents to the circle at points A, B, C, D, E, F.
Tangents intersect at points P, Q, R, S, T, U.

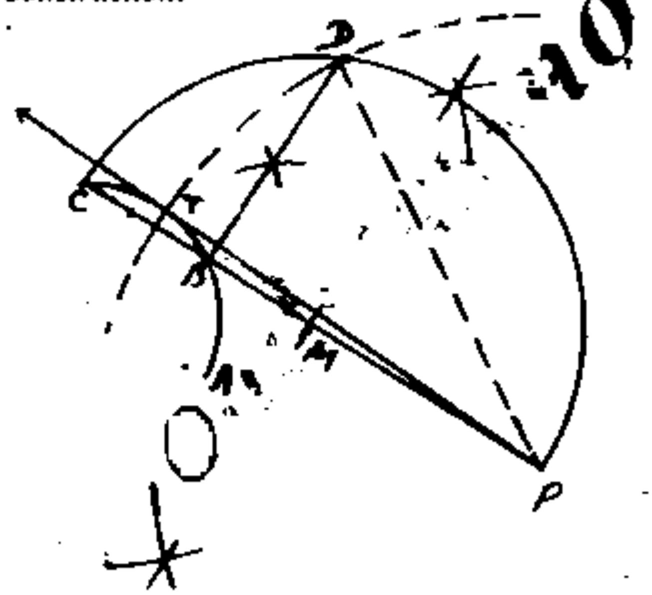
Result:

$PQRSTU$ is the required hexagon.

EXERCISE 13.3

1. ***In an arc ABC the length of the chord |BC| = 2cm. Draw a secant |PBC| = 8cm, where P is the point outside the arc. Draw a tangent through point P to the arc.***

Steps of Construction:



(i) Draw an arc $\widehat{ABC}$

(ii) Take a chord $\overline{BC} = 2$ cm.

(iii) Produce $\overline{CB}$ towards B and take point P that $\overline{PBC}$ secant is 8 cm.

(iv) Find M, the mid point of $\overline{CP}$.

(v) Take M as centre and draw a semi circle.

(vi) Draw $\overline{DB} \perp \overline{CP}$ which meets the semi circle at point D.

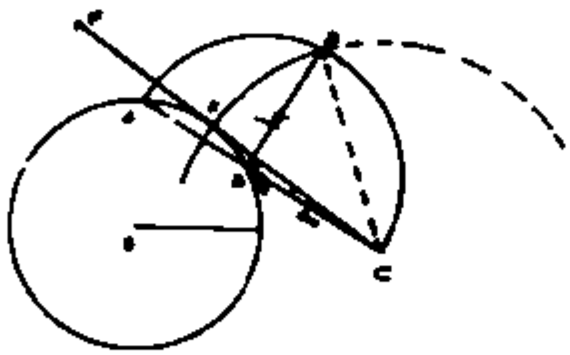
(vii) Take P as centre and draw an arc of radius $m\overline{PD}$, this arc intersect the given arc at T.

(viii) Join P to T and produce it.

Result:

$\overrightarrow{PT}$ is the required tangent.

2. ***Construct a circle with diameter 8cm. Indicate a point C, 5cms away from its circumference. Draw a tangent from point C to the circle without using its centre.***



Steps of Construction:

(i) Draw a circle of radius $\frac{8}{2} = 4cm$ with centre at O.

(ii) Take a secant $\overline{ABC}$ such that point C is 5cm away from circumference of the circle.

(iii) Find M, the mid point of $\overline{AC}$.

(iv) Draw a semi-circle of radius $|\overline{AM}| = |\overline{CM}|$ with centre at M.

(v) Draw a perpendicular at point B which meets the semi-circle at D.

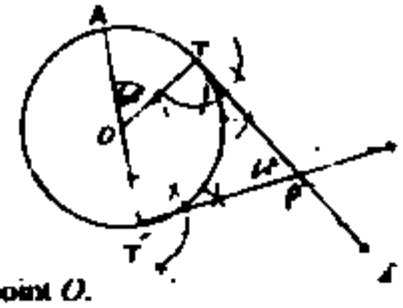
(vi) Draw an arc of radius $|\overline{CD}|$ with centre at C. This arc cuts the given circle at point E.

(vii) Join C with E.

RESULT: $\overrightarrow{CEF}$ is the required tangent.

3. *Construct a circle of radius 2cm. Draw two tangents making an angle of 60° with each other.*

Steps of Construction:



(i) Take a point O.

(ii) Take O as centre and draw a circle with radius 2 cm.

(iii) Draw $\overline{AOT'}$ any diameter.

(iv) Draw $\angle AOT = 60^{\circ}$.

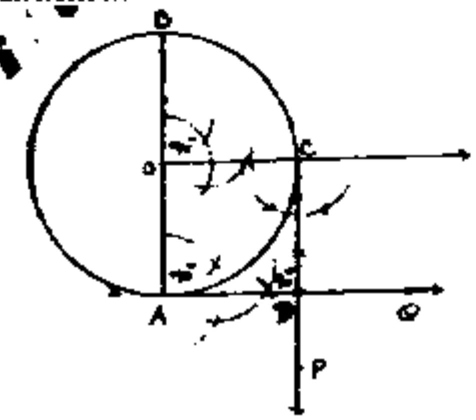
(v) Draw $\overrightarrow{TP}$ and $\overline{T'P}$ tangents at T, T', that intersect at P.

Result:

$\overrightarrow{TP}$ and $\overrightarrow{T'P}$ are the required tangents.

4. *Draw two perpendicular tangents to a circle of radius 3cm.*

Steps of Construction:



(i) Take a point O.

(ii) Take O as centre and a circle of radius 3 cm.

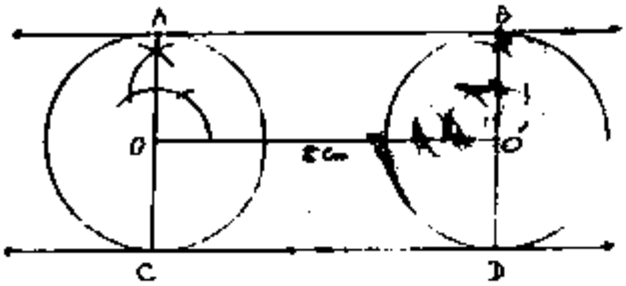
(iii) Take $\overline{AOB}$ any diameter of the circle.
(iv) Draw $m\angle BOC = 90^0$.
(v) Draw tangents at point A, C.

These are $\overrightarrow{CP}$, $\overrightarrow{AQ}$

Result:

$\overrightarrow{AQ}$, $\overrightarrow{CP}$ are the required tangents that meet at point D at 90^0.

5. ***Two equal circles are at 8cm apart. Draw two direct common tangents of this pair of circles.***



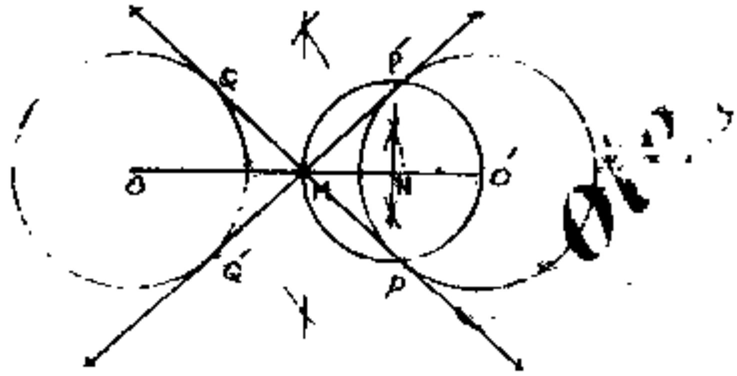
Steps of Construction:

(i) Draw $\overline{OO'} = 8$ cm.
(ii) Draw two circles of equal size on O and O'.
(iii) Draw $\overline{OA} \perp \overline{OO'}$ and produce towards O. $\overline{OA}$ produced meets the circle at C.
(iv) Draw $\overline{O'B} \perp \overline{OO'}$ and produce it towards O'. $\overline{BO'}$ produced meets the circle at D.
(v) Join A with B and produce it both sides.
(vi) Join C with D and produce both sides.

Result:

$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are the common external tangents.

6. ***Draw two equal circles of each radius 2.4cm. If the distance between their centres is 7cm then draw their transverse tangents.***



Steps of Construction:

(i) Draw $m\overline{OO'} = 7$ cm.

(ii) Draw two circles of 2.4cm radius on O and O′.

(iii) Find M, the mid point of $\overline{OO'}$.

(iv) Find N, the mid point of $\overline{MO'}$.

(v) Draw a circle with centre at N and of radius $\overline{NO'}$. This circle intersects the circle at P and P′.

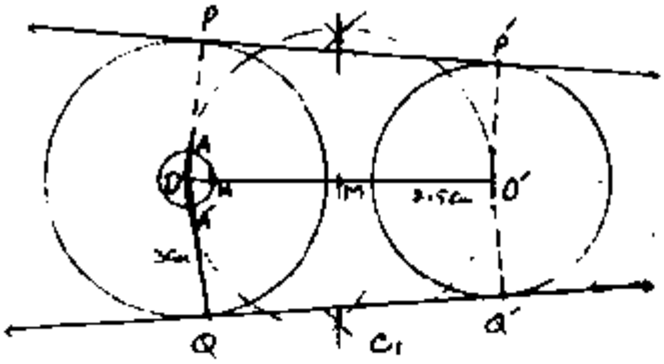
(vi) Join P' with M and produce towards M, it touch the second circle at Q'.

(vii) Join P with M and produce towards M.

$\overline{PM}$ produced touches the second circle at Q.

$\overleftrightarrow{PQ}$ $\overleftrightarrow{P'Q'}$ are the required tangents.

7. ***Draw two circles with radii 2.5cm and 3cm, of their centres are 6.5cm apart then draw two direct common tangents.***



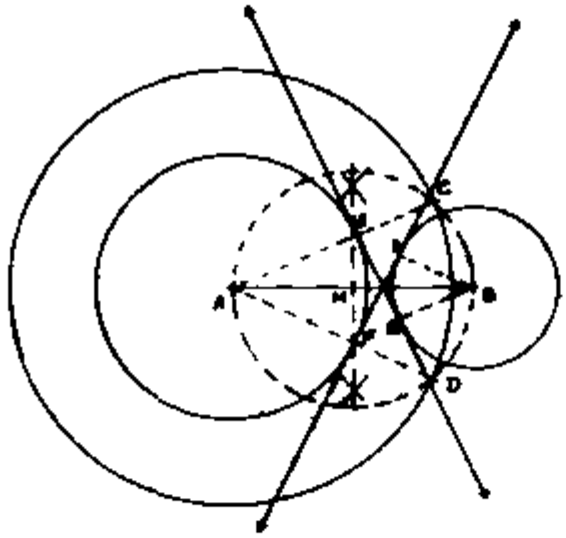
Steps of Construction:

(i) . Draw $\overline{OO'}$ of length 6.5 cm.
(ii) Take O as centre and draw a circle with radius 3 cm.
(iii) Take O' as centre and draw a circle with radius 2.5 cm.
(iv) Find M, mid-point of $\overline{OQ'}$. Take M as centre and draw a circle with radius $m\overline{MO'}$.
(v) Cut $m\overline{ON} = 3 - 2.5 = .5$ cm and take O as centre, draw the circle with radius $m\overline{ON}$. This circle intersects the circle C_1 at point A, A'.
(vi) Join O with A, A' and produce on both sides. $\overline{OA}$ and $\overline{OA'}$ produced intersect the larger circle at P and Q.
(vii) Draw $\overline{O'P'} \parallel \overline{OP}$ and $\overline{O'Q'} \parallel \overline{OQ}$
(viii) Join P with P' and Q with Q'.

Result:

$\overleftrightarrow{PP'}$ and $\overleftrightarrow{QQ'}$ are the required tangents.

8. ***Draw two circles with radii 3.5cm and 2cm of their centres are 6cm apart then draw two transverse common tangents.***



Ans. ***Construction:***

(i) Take a line segment of measure $\overline{AB} = 6cm$.

(ii) Draw two circles of radii 3.5 and $2cm$ with centres at A and B respectively.

(iii) Taking A as centre draw a circle of radius $3.5 + 2 = 5.5\ cm$.

(iv) Bisect the line segment $\overline{AB}$ at point M.

(v) Take M as centre and draw a circle of radius $\overline{MA}$ which intersects the big circle at points C and D.

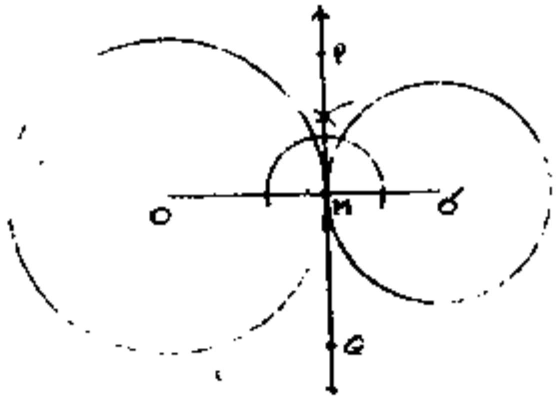
(vi) Join A with C and D to produce $\overline{AD}$ and $\overline{AC}$.
$\overline{AD}$ and $\overline{AC}$ meet the inner circle at E and F.

(vii) Draw $\overline{BQ} \parallel \overline{AE}$ and $\overline{BP} \parallel \overline{AF}$.

(viii) Join E with Q and produce on both sides.
Join F with P and produce on both sides.

Result: $\overleftrightarrow{EQ}$ and $\overleftrightarrow{FP}$ are the required tangents.

9. ***Draw two common tangents to two touching circles of radii 2.5cm and 3.5cm.***



Steps of Construction:

(i) Draw a line segment $\overline{OO'}$ of measure $2.5 + 3.5 = 6.0$ cm

(ii) Take O as centre and draw a circle with radius $m\overline{OM}$ 3.5 cm.

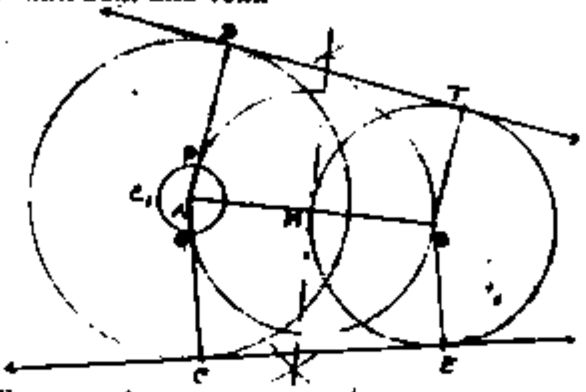
(iii) Take O' as centre and draw a circle with radius 2.5 cm. These circle touch each other at point M.

(iv) Draw $\overleftrightarrow{PQ} \perp OO'$.

Result:

$\overleftrightarrow{PQ}$ is the required common tangents.

10. Draw two common tangents to two intersecting circle of radii 3cm and 4cm.



Steps of Construction:

(i) Take a line segment $\overline{AB}$ that $m\overline{AB} = 3 + 4 = 7$ cm.
(ii) Draw two circles of radii 4cm, 3cm with centres at A, B.
(iii) Taking A as centre draw a circle with radius $4 - 3 = 1$ cm.
(iv) Bisect the line segment $\overline{AB}$ at point M.
(v) Take M as centre and draw a circle of radius $m\overline{MB}$, this circle intersects. Circle C_1 at P, Q.
(vi) Join A with P and Q and produce $\overline{AP}$, $\overline{AQ}$ to meet the larger circle at D, C.
(vii) Draw $\overline{BT} \parallel \overline{AD}$ and $\overline{BE} \parallel \overline{AC}$.
(viii) Join D with T and produce both sides.
(ix) Join C with E and produce both sides.

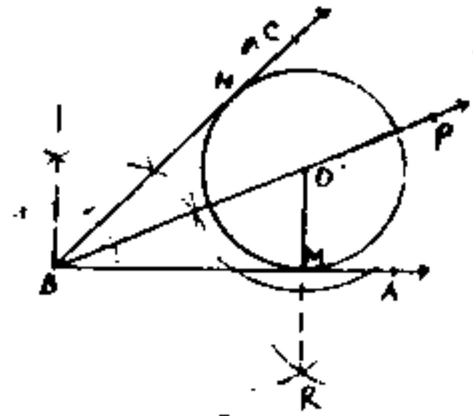
Result:

$\overleftrightarrow{DT}$ and $\overleftrightarrow{CE}$ are the required tangents.

11. Draw circles which touches both the arms of angles (i) 45^O (ii) 60^O.

11. (i)

Steps of Construction:



(i) Draw an angle ABC of 45^{o}.

(ii) Draw $\overrightarrow{BP}$ bisector of angle $\angle ABC$.

(iii) Take any point O on $\overrightarrow{BP}$.

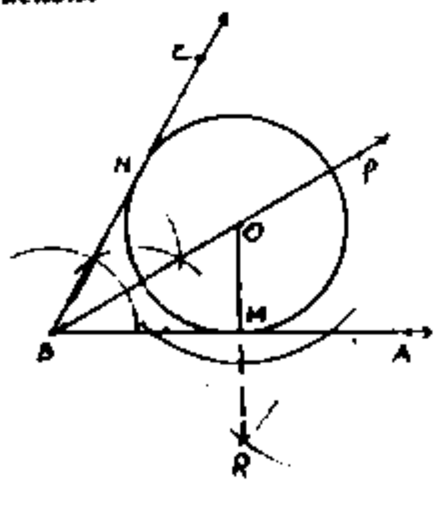
(iv) Drop $\overline{OM} \perp \overline{BA}$.

(v) Take O as centre and draw a circle with radius $m\overline{OM}$

This circle touches arm $\overrightarrow{BC}$ at N also.

11.(ii)

Steps of Construction:



(i) Draw an angle $\angle ABC$ of 60°.

(ii) Draw $\overrightarrow{BP}$ bisector of angle $\angle ABC$.

(iii) Take any point O on $\overrightarrow{BP}$

(iv) Drop $\overline{OM} \perp \overline{BA}$.

(v) Take O as centre and draw a circle with radius $m\overline{OM}$.

This circle touches arm $\overline{BC}$ at N also.

MISCELLANEOUS EXERCISE 13

I. ***Multiple Choice Questions***

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) The circumference of a circle is called

(a) chord (b) segment (c) boundary

(ii) A line intersecting a circle is called

(a) tangent (b) secant (c) chord

(iii) The portion of a circle between two radii and an arc is called

(a) sector (b) segment (c) chord

(iv) Angle inscribed in a semi-circle is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$

(v) The length of the diameter of a circle is how many times the radius of the circle

(a) 1 (b) 2 (c) 3

(vi) The tangent and radius of a circle at the point of contact are

(a) parallel
(b) not perpendicular
(c) perpendicular

(vii) Circles having three points in common

(a) over lapping
(b) collinear

(c) not coincide

(viii) If two circles touch each other, their centres and point of contact are
(a) coincident
(b) non-collinear
(c) collinear

(ix) The measure of the external angle of a regular hexagon is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$

(x) If the incentre and circumcentre of a triangle coincide, the triangle is
(a) an isoscenes
(b) a right triangle
(c) an equilateral

(xi) The measure of the external angle of a regular octagon is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{8}$

(xii) Tangents drawn at the end points of the diameter of a circle are
(a) parallel (b) perpendicular (c) Intersecting

(xiii) The lengths of two transverse tangents to a pair of circles are
(a) unequal (b) equal (c) overlapping

(xiv) How many tangents can be drawn from a point outside the circle?
(a) 1 (b) 2 (c) 3

(xv) If the distance between the centres of two circles is equal to the sum of their radii, then the circles will
(a) intersect
(b) do not intersect
(c) touch each other externally

(xvi) If the two circles touches externally, then the distance between their centers
(a) difference of their radii
(b) sum of their radii
(c) product of their radii

(xvii) How many common tangents can be drawn for two touching circles?
(a) 2 (b) 3 (c) 4

(xviii) How many common tangents can be drawn for two disjoint circles?
(a) 2 (b) 3 (c) 4

Answers:

(i)	c	(ii)	b	(iii)	a	(iv)	a	(v)	b
(vi)	c	(vii)	a	(viii)	c	(ix)	a	(x)	c
(xi)	a	(xii)	a	(xiii)	b	(xiv)	b	(xv)	a
(xvi)	b	(xvii)	b	(xviii)	c				

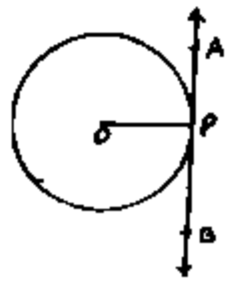
2. ***Write short answers of the following questions***

(i) Define and draw the following geometric figures:

(a) The segment of a circle.

The area contained between a chord and the arc which it cuts off is called a **segment** of the circle.

(b) The tangent to a circle.

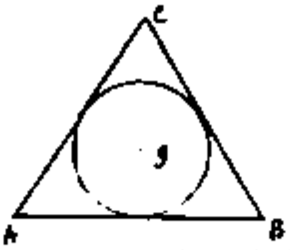


A line that touch a circle is called its tangent. $\overleftrightarrow{APB}$ is tangent to the circle with centre O at point P.

(c) The sector of a circle.

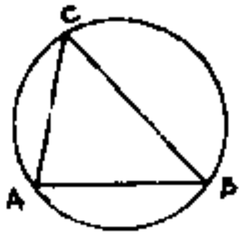
The area contained between two radii and the arc of the circle which they intercept, is called a sector of the circle.

(d) The inscribed circle.



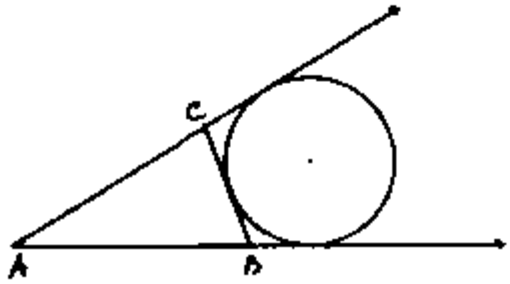
The sides of triangle touch a circle internally, such a circle is called inscribed circle of the triangle.

(e) The circumscribed circle.



A circle that passes through the vertices of a triangle is called circumscribed circle of the triangle.

(f) The escribed circle.



A circle that touches two sides of a triangle internally and one side externally is called an escribed circle.

(ii) The length of each side of a regular octagon is 3cm. Measure its perimeter.

Solution: Length of side = 3 cm

Number of sides of an octagon = 8

Perimeter = 3×8
= 24 cm

(iii) Write down the formula for finding the angle subtended by the side if a n-sided polygon at the centre of the circle.

Ans. $\frac{360^0}{n}$

(iv) The length of the side of a regular pentagon is 5cm what is its perimeter?

Solution: Length of each side = 5 cm
Number of sides = 5
Perimeter = 5×5
= 25 cm

3. Fill in the blanks.

(i) The boundary of a circle is called ______.

(ii) The circumference of a circle is called ______.

(iii) The line joining the two points of circle is called ______.

(iv) The point of intersection of perpendicular bisectors of two non-parallel chords of a circle is called the ______.

(v) Circle having three points in common will ______.

(vi) The distance of a point inside the circle from its centre is ______ than the radius.

(vii) The distance of a point outside the circle from its centre is ______ than the radius.

(viii) A circle has only ______ centre.

(ix) One and only one circle can be drawn through three ______ points.

(x) Angle inscribed in a semi-circle is a ______ angle.

(xi) If two circles touch each other, the point of ______ and their ______ are collinear.

(xii) If two circles touch each other, their point of contact and centres are ______.

(xiii) From a point outside the circle ______ tangents can be drawn.

(xiv) A tangent is ______ to the radius of a circle at its point of contact.

(xv) The straight line drawn ⊥ to the radius of a circle is called the ______ to the circle.

(xvi) Two circles can not cut each other at more than ______ points.

(xvii) The ⊥ bisector of a chord of a circle passes through the ______.

(xviii) The length of two direct common tangents to two circles are ______ to each other.

(xix) The length of two transverse common tangents to two circles are ______ to each other.

(xx) If the in-centre and circum-center of a triangle coincide the triangle is ______.

(xxi) Two intersecting circles are not ______.

(xxii) The centre of an inscribed circle is called ______.

(xxiii) The centre of a circumscribed circle is called ______.

(xxiv) The radius of an inscribed circle is called ______.

(xxv) The radius of a circumscribed circle is called ______.

Answers:

(i)	circumference	(ii)	boundary
(iii)	chord	(iv)	centre
(v)	coincide	(vi)	less
(vii)	greater	(viii)	one
(ix)	non-collinear	(x)	right
(xi)	contact, centres	(xii)	collinear
(xiii)	two	(xiv)	perpendicular
(xv)	tangent	(xvi)	two
(xvii)	centre	(xviii)	equal
(xix)	equal	(xx)	equilateral
(xxi)	concentric	(xxii)	incentre
(xxiii)	circumcentre	(xxiv)	in-radius
(xxv)	circum-radius		